

Sliding Control of MIMO Nonlinear Systems

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Abstract. Sliding control of MIMO (multivariable input-multivariable output) nonlinear *minimum phase* systems is discussed. Stability conditions related to model errors are emphasized. Global asymptotic stability is guaranteed by applying Barbalat's Lyapunov-like lemma. The control law is applied to a simulator of a polymerization reactor.

Keywords. Feedback Linearization, Sliding Control, Lyapunov Stability Theory, Polymerization Reactor.

1 INTRODUCTION

The major problem with feedback linearization techniques is robustness due to imprecise cancellations of the model nonlinearities. In the case of parametric uncertainties, global asymptotically stable controllers may be found by using the Lyapunov stability theory. Well known techniques, arising from this approach are adaptive feedback linearization and sliding control which both have been successfully applied in robotic control [5]. Sliding control has also been applied in the control of underwater vehicles which are highly nonlinear and time-varying in their parameters [7]. This paper shows how sliding control can be applied to MIMO nonlinear *minimum phase* systems in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u}$ with $\mathbf{y} = \mathbf{h}(\mathbf{x})$. These results are also extended to the more general model class $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ with $\mathbf{y} = \mathbf{h}(\mathbf{x})$. In some cases, the general model class is necessary because of the complex structure of a model. This is true in certain process control applications, for instance. An example from a polymerization plant will be used in this investigation.

The paper is outlined as follows: Section 2 discusses feedback linearization and sliding control of MIMO affine systems. Sliding control of the more general model class is examined in Section 3, while results from the simulation studies are presented in Section 4. The paper ends with the conclusions.

2 AFFINE SYSTEMS

Before we discuss the more general nonlinear model class $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ we will consider MIMO nonlinear systems which are linear in control or affine. These systems can be expressed as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \quad (1)$$

with $(\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, \mathbf{u} \in \mathbb{R}^p)$ and $(\mathbf{f}(\mathbf{x}), \mathbf{G}(\mathbf{x}))$

and $\mathbf{h}(\mathbf{x})$ smooth.

2.1 Review of Input-Output Feedback Linearization of Affine Systems

Differentiating the output \mathbf{y} with respect to time yields

$$\dot{\mathbf{y}} = \mathbf{H}_x(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{H}_x(\mathbf{x})\mathbf{G}(\mathbf{x})\mathbf{u}$$

where $\mathbf{H}_x(\mathbf{x}) = \frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}$ is the Jacobian. This control problem may be reduced to that of controlling the linear system

$$\dot{\mathbf{y}} = \mathbf{v}$$

where the choice: $\mathbf{v} = \dot{\mathbf{y}}_d - \mathbf{K}_P(\mathbf{y} - \mathbf{y}_d)$ yields the error equation

$$\dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} = 0 \quad (2)$$

Here \mathbf{K}_P is a positive definite design matrix of appropriate dimension, \mathbf{y}_d is the desired trajectory and $\mathbf{e} = \mathbf{y} - \mathbf{y}_d$. If the system Eq. 1 is square i.e. $m = p$, and $\mathbf{H}_x(\mathbf{x})\mathbf{G}(\mathbf{x})$ is nonsingular for all $\mathbf{x} \in \mathbb{R}^n$ the actual control input \mathbf{u} could be calculated as

$$\mathbf{u} = (\mathbf{H}_x(\mathbf{x})\mathbf{G}(\mathbf{x}))^{-1} [\mathbf{v} - \mathbf{H}_x(\mathbf{x})\mathbf{f}(\mathbf{x})]$$

It is convenient to let r_j denote the systems relative degree i.e. the smallest number of differentiations r_j the output y_j has to be differentiated for one of the control inputs to appear. The total relative degree is defined as $r = r_1 + \dots + r_m$. Let us consider the case of $r < n$. Defining $\mathbf{G} = [\mathbf{g}_1, \dots, \mathbf{g}_p]$ and $L_{\mathbf{f}}h_j$ and $L_{\mathbf{g}_i}h_j$ as the Lie derivatives of h_j with respect to \mathbf{f} and \mathbf{g}_i , respectively, implies that Eq. 1 can be expressed as [2]

$$y_j^{(r_j)} = L_{\mathbf{f}}^{r_j} h_j + \sum_{i=1}^m L_{\mathbf{g}_i} (L_{\mathbf{f}}^{r_j-1} h_j) u_i \quad (3)$$

The smallest integer r_j is found by requiring that at least one of the Lie derivatives $L_{\mathbf{g}_i} (L_{\mathbf{f}}^{r_j-1} h_j) \neq 0 \forall \mathbf{x}$.

Notice that if the control input does not appear after at least r_j differentiations the system will not be controllable. Let the $m \times m$ decoupling matrix $\mathbf{G}^*(\mathbf{x})$ be defined as [2]

$$\mathbf{G}^*(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{g}_1}(L_{\mathbf{f}}^{r_1-1}h_1) & \dots & L_{\mathbf{g}_m}(L_{\mathbf{f}}^{r_1-1}h_1) \\ \vdots & & \vdots \\ L_{\mathbf{g}_1}(L_{\mathbf{f}}^{r_m-1}h_m) & \dots & L_{\mathbf{g}_m}(L_{\mathbf{f}}^{r_m-1}h_m) \end{bmatrix}$$

and

$$\mathbf{f}^*(\mathbf{x}) = (L_{\mathbf{f}}^{r_1}h_1, \dots, L_{\mathbf{f}}^{r_m}h_m)^T$$

Hence, Eq. 3 can be written in a compact form as

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \mathbf{f}^*(\mathbf{x}) + \mathbf{G}^*(\mathbf{x}) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad (4)$$

The equivalent linear system to be controlled is

$$y^{(r_j)} = v_j \quad , \quad j = 1..m \quad (5)$$

If $\mathbf{G}^*(\mathbf{x})$ is non-singular the nonlinear feedback control law

$$\mathbf{u} = (\mathbf{G}^*(\mathbf{x}))^{-1} [\mathbf{v} - \mathbf{f}^*(\mathbf{x})] \quad (6)$$

yields the decoupled system Eq. 5 directly. A system with relative degree (r_1, \dots, r_m) can be transformed into a so-called *normal form* by applying a diffeomorphism $(\zeta, z) = T(\mathbf{x})$ defined as

$$\begin{aligned} \zeta_1^1 &= h_1 & \zeta_2^1 &= L_{\mathbf{f}}h_1 & \dots & \zeta_{r_1}^1 &= L_{\mathbf{f}}^{r_1-1}h_1 \\ &\vdots & & & & & \vdots \\ \zeta_1^m &= h_m & \zeta_2^m &= L_{\mathbf{f}}h_m & \dots & \zeta_{r_m}^m &= L_{\mathbf{f}}^{r_m-1}h_m \end{aligned}$$

This implies that Eq. 4 can be expressed as

$$\begin{aligned} \dot{\zeta}_1^j &= \zeta_2^j \\ &\vdots \\ \dot{\zeta}_{r_j}^j &= f_j^*(\zeta, z) + \sum_{i=1}^m \mathbf{G}_{ji}^*(\zeta, z)u_i \\ y_j &= \zeta_1^j \end{aligned}$$

where $j = 1, 2, \dots, m$, and

$$\dot{z} = \psi(\zeta, z) + \Psi(\zeta, z)\mathbf{u} \quad (7)$$

The state vector \mathbf{z} denotes the internal dynamics and

$$\begin{aligned} \psi(\zeta, z)_k &= L_{\mathbf{f}}\zeta_k(\mathbf{x}) \\ \Psi_{ki}(\zeta, z) &= L_{\mathbf{g}_i}\zeta_k(\mathbf{x}) \end{aligned}$$

where $(k = 1, \dots, n - r)$ and $(i = 1, \dots, m)$. The zero-dynamics of the nonlinear system is defined as

the dynamics of the system when the outputs are constrained to be identically zero i.e. $\zeta = 0$. This is obtained by choosing the control inputs as

$$\mathbf{u}(0, z) = -(\mathbf{G}^*(0, z))^{-1} \mathbf{f}^*(0, z)$$

Eliminating \mathbf{u} from Eq. 7 yields the zero dynamics

$$\dot{z} = \psi(0, z) - \Psi(0, z)(\mathbf{G}^*(0, z))^{-1} \mathbf{f}^*(0, z)$$

Notice that the zero dynamics are made unobservable by state feedback. The nonlinear system Eq. 1 is said to be *non-minimum phase* if the zero dynamics are unstable and *asymptotically minimum phase* if the zero dynamics are asymptotically stable [1]. For minimum phase systems, feedback linearization results in bounding tracking if the desired motion trajectory is bounded. The proof is given in [2].

Uncertainties in the functions \mathbf{f} and \mathbf{G} in the nonlinear decoupling may result in imprecise cancellations of the nonlinear terms. In the next section we will demonstrate how sliding control can be used to compensate for model uncertainties.

2.2 Sliding Control of Affine Systems

The sliding design methodology is described in detail by e.g. [4, 6, 8]. In the following, it is convenient to define the operators:

$$\begin{aligned} |\mathbf{x}| &= [|x_1|, |x_2|, \dots, |x_n|]^T \\ \mathbf{sgn}(\mathbf{x}) &= [\mathbf{sgn}(x_1), \mathbf{sgn}(x_2), \dots, \mathbf{sgn}(x_n)]^T \\ \mathbf{x} \times \mathbf{y} &= [x_1 y_1, x_2 y_2, \dots, x_n y_n]^T \end{aligned}$$

In sliding control the error equation, Eq. 2, is replaced by

$$\dot{\mathbf{s}} + \mathbf{k} \times \mathbf{sgn}(\mathbf{s}) = \mathbf{w}(t)$$

where \mathbf{s} is a measure of tracking and $\mathbf{w}(t)$ represents the external signals. According to the above definitions, $\mathbf{k} \times \mathbf{sgn}(\mathbf{s})$ yields a vector of elements $k_i \mathbf{sgn}(s_i)$. If all k_i satisfy $k_i > \sup_t |w_i(t)|$, then \mathbf{s} goes to zero in finite time. Define a sliding surface c.f. [5]

$$s_j = \left(\frac{d}{dt} + \lambda_j \right)^{r_j-1} e_j \quad \text{where} \quad e_j = y_j - y_{j,d} \quad (8)$$

where λ_j is a positive scalar specifying the control bandwidth and $y_{j,d}$ is the desired trajectory. For systems of relative degree $r_j = 1$, Eq. 8 simply yields $s_j = e_j$. Define a virtual reference vector α_d such that

$$\dot{s}_j = y_j^{(r_j)} - \alpha_{d,j} \quad , \quad j = 1, \dots, m \quad (9)$$

For the system, Eq. 4, we propose a control law

$$\mathbf{u} = (\hat{\mathbf{G}}^*(\mathbf{x}))^{-1} [\alpha_d - \hat{\mathbf{f}}^*(\mathbf{x}) - \mathbf{k} \times \mathbf{sgn}(\mathbf{s})] \quad (10)$$

where the hat denotes the estimates of the nonlinear functions. The bounds of the elements k_i may be derived by applying Barbalat's lemma. Notice, that the existence of \mathbf{u} is influenced by the choice of the controlled variables \mathbf{y} i.e. the existence of the inverse of $\mathbf{G}^*(\mathbf{x})$. Assume that the parametric uncertainties in the nonlinear functions \mathbf{f}^* and \mathbf{G}^* satisfy the following bounds

$$\begin{aligned} |\hat{f}_j^*(\mathbf{x}) - f_j^*(\mathbf{x})| &\leq \delta_i \quad (11) \\ \mathbf{G}^*(\mathbf{x}) &= (\mathbf{I} + \mathbf{\Delta})\hat{\mathbf{G}}^*(\mathbf{x}) \quad , \quad |\Delta_{ij}| \leq D_{ij} \end{aligned}$$

where $(i = 1, \dots, m)$, $(j = 1, \dots, m)$ and $\bar{\sigma}(\mathbf{\Delta}) < 1$. Consider a Lyapunov-like function candidate

$$V(\mathbf{s}, t) = \frac{1}{2} \mathbf{s}^T \mathbf{s} \quad (12)$$

Differentiating $V(\mathbf{s}, t)$ with respect to time and substituting Eqs. 4, 9, 10 and 11 yields

$$\begin{aligned} \dot{V} &= \mathbf{s}^T \dot{\mathbf{s}} = \mathbf{s}^T \left[(\mathbf{f}^* - \hat{\mathbf{f}}^*) + \mathbf{\Delta}(\boldsymbol{\alpha}_d - \hat{\mathbf{f}}^*) \right. \\ &\quad \left. - (\mathbf{I} + \mathbf{\Delta}) \mathbf{k} \times \text{sgn}(\mathbf{s}) \right] \end{aligned}$$

From this it is seen that if $k_i \geq k'_i \quad \forall \quad i$ where \mathbf{k}' satisfies

$$(\mathbf{I} - \bar{\mathbf{D}}) \mathbf{k}' = \boldsymbol{\delta} + \mathbf{D} |\boldsymbol{\alpha}_d - \hat{\mathbf{f}}^*(\mathbf{x})| + \boldsymbol{\eta} \quad (13)$$

where \mathbf{D} is an $m \times m$ matrix with elements D_{ij} and $\bar{\mathbf{D}}$ is

$$\bar{\mathbf{D}} = \begin{bmatrix} D_{11} & -D_{12} & \dots & -D_{1m} \\ -D_{21} & D_{22} & & -D_{2m} \\ \vdots & & \ddots & \vdots \\ -D_{m1} & -D_{m2} & \dots & D_{mm} \end{bmatrix} \quad (14)$$

the sliding condition

$$\dot{V} = -\boldsymbol{\eta}^T |\mathbf{s}| = \sum_{i=1}^m -\eta_i |s_i| \leq 0 \quad , \quad \eta_i > 0 \quad (15)$$

Hence, applying Barbalat's Lyapunov-like lemma ensures that $\mathbf{s} \rightarrow 0$ and thus $\mathbf{e} \rightarrow 0$.

3 SYSTEMS IN THE FORM:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad \text{WITH} \quad \mathbf{y} = \mathbf{h}(\mathbf{x})$$

Although Eq. 1 is a fairly general model description, not all systems can be modeled using this structure. For instance consider the following model class

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}_1(\mathbf{x})\mathbf{u} + \mathbf{G}_2(\mathbf{u})\mathbf{x}$$

which is nonlinear in the control input \mathbf{u} . This justifies the use of a more general model class

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \quad (16)$$

Such a system will also be discussed in Section 4 in conjunction with the simulation of a polymerization reactor.

3.1 Input-Output Linearization

For simplicity let us first consider a system where $m = p$. Differentiating \mathbf{y} in Eq. 16 with respect to time yields

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{H}_x \mathbf{f} \\ \ddot{\mathbf{y}} &= \frac{\partial}{\partial \mathbf{x}} (\mathbf{H}_x \mathbf{f}) \mathbf{f} + \frac{\partial}{\partial \mathbf{u}} (\mathbf{H}_x \mathbf{f}) \dot{\mathbf{u}} \end{aligned} \quad (17)$$

The nonlinear control law (assuming that $\frac{\partial}{\partial \mathbf{u}} (\mathbf{H}_x \mathbf{f})$ is non-singular)

$$\dot{\mathbf{u}} = \left[\frac{\partial}{\partial \mathbf{u}} (\mathbf{H}_x \mathbf{f}) \right]^{-1} \left[\mathbf{v} - \frac{\partial}{\partial \mathbf{x}} (\mathbf{H}_x \mathbf{f}) \mathbf{f} \right]$$

yields the linear system

$$\ddot{\mathbf{y}} = \mathbf{v}$$

which suggests \mathbf{v} to be chosen as

$$\mathbf{v} = \ddot{\mathbf{y}}_d - \mathbf{K}_D \dot{\mathbf{e}} - \mathbf{K}_P \mathbf{e}$$

where $\mathbf{e} = \mathbf{y} - \mathbf{y}_d$ and \mathbf{K}_D and \mathbf{K}_P are design matrices of appropriate dimensions chosen such that the error equation $\ddot{\mathbf{e}} + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} = 0$ is stable. In the general case, let γ_j be the smallest number of differentiations of the output y_j for one of the derivatives \dot{u}_i to appear, then

$$y_j^{(\gamma_j)} = L_{\mathbf{f}}^{\gamma_j} h_j + \frac{\partial}{\partial \mathbf{u}} \left(L_{\mathbf{f}}^{\gamma_j - 1} h_j \right) \dot{\mathbf{u}} \quad , \quad j = 1, \dots, m$$

or equivalently

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \bar{\mathbf{f}}(\mathbf{x}, \mathbf{u}) + \bar{\mathbf{G}}(\mathbf{x}, \mathbf{u}) \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_m \end{bmatrix}$$

where $\bar{f}_j(\mathbf{x}, \mathbf{u}) = L_{\mathbf{f}}^{\gamma_j} h_j(\mathbf{x})$ and

$$\bar{\mathbf{G}}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial}{\partial u_1} \left(L_{\mathbf{f}}^{\gamma_1 - 1} h_1 \right) & \dots & \frac{\partial}{\partial u_m} \left(L_{\mathbf{f}}^{\gamma_1 - 1} h_1 \right) \\ \vdots & & \vdots \\ \frac{\partial}{\partial u_1} \left(L_{\mathbf{f}}^{\gamma_m - 1} h_m \right) & \dots & \frac{\partial}{\partial u_m} \left(L_{\mathbf{f}}^{\gamma_m - 1} h_m \right) \end{bmatrix}$$

The nonlinear control law (assuming that $\bar{\mathbf{G}}(\mathbf{x}, \mathbf{u})$ is non-singular)

$$\dot{\mathbf{u}} = \left(\bar{\mathbf{G}}(\mathbf{x}, \mathbf{u}) \right)^{-1} \left[\mathbf{v} - \bar{\mathbf{f}}(\mathbf{x}, \mathbf{u}) \right]$$

yields $y_j^{(\gamma_j)} = v_j$ for $j = 1, \dots, m$.

3.2 Sliding Control

The corresponding sliding control law is found by defining a sliding surface

$$s_j = \left(\frac{d}{dt} + \lambda_j \right)^{\gamma_j - 1} e_j \quad \text{where} \quad \gamma_j > 1$$

Let $\mathbf{A} = \text{diag}\{\lambda_j\}$. Hence, \dot{V} can be written as

$$\dot{V} = \mathbf{s}^T \dot{\mathbf{s}} = \mathbf{s}^T (\bar{\mathbf{f}}(\mathbf{x}, \mathbf{u}) + \bar{\mathbf{G}}(\mathbf{x}, \mathbf{u})\dot{\mathbf{u}} - \boldsymbol{\alpha}_d)$$

This suggests the control law

$$\dot{\mathbf{u}} = \hat{\mathbf{G}}^{-1}(\mathbf{x}, \mathbf{u}) \left[\boldsymbol{\alpha}_d - \hat{\mathbf{f}}(\mathbf{x}, \mathbf{u}) - \mathbf{k} \times \text{sgn}(\mathbf{s}) \right] \quad (18)$$

where the parametric uncertainties in the nonlinear functions are assumed to satisfy the following bounds

$$\begin{aligned} |\hat{\mathbf{f}}_j(\mathbf{x}, \mathbf{u}) - \bar{\mathbf{f}}_j(\mathbf{x}, \mathbf{u})| &\leq \delta_j \\ \bar{\mathbf{G}}(\mathbf{x}, \mathbf{u}) &= (\mathbf{I} + \boldsymbol{\Delta})\hat{\mathbf{G}}(\mathbf{x}, \mathbf{u}) \quad , \quad |\Delta_{ij}| \leq D_{ij} \end{aligned}$$

where $(i = 1, \dots, m)$, $(j = 1, \dots, m)$ and $\bar{\sigma}(\boldsymbol{\Delta}) < 1$. Hence, \dot{V} can be written as

$$\begin{aligned} \dot{V} = \mathbf{s}^T &\left[-(\mathbf{I} + \boldsymbol{\Delta}) \mathbf{k} \times \text{sgn}(\mathbf{s}) + (\bar{\mathbf{f}} - \hat{\mathbf{f}}) \right. \\ &\left. + \boldsymbol{\Delta} (\boldsymbol{\alpha}_d - \hat{\mathbf{f}}) \right] \end{aligned}$$

From the expression for \dot{V} it is easily seen that \mathbf{k} must satisfy $k_i \geq k'_i \quad \forall \quad i$ where:

$$(\mathbf{I} - \bar{\mathbf{D}})\mathbf{k}' = \boldsymbol{\delta} + \mathbf{D} \left| \boldsymbol{\alpha}_d - \hat{\mathbf{f}}(\mathbf{x}, \mathbf{u}) \right| + \boldsymbol{\eta}$$

with the matrix $\bar{\mathbf{D}}$ defined as in Eq. 14 to achieve $\dot{V} \leq -\boldsymbol{\eta}^T \mathbf{s} \mid \mathbf{s} \mid \leq 0$ which according to Barbalat's lemma implies that $\mathbf{s} \rightarrow 0$ and thus $\mathbf{e} \rightarrow 0$. Notice that the requirement that both \mathbf{y} and $\dot{\mathbf{y}}$ must be measured to calculate the measure of tracking \mathbf{s} is quite restrictive, especially in cases with noise on \mathbf{y} . If $\dot{\mathbf{y}}$ is not available from measurements, some algorithm for approximating $\dot{\mathbf{y}}$ must be used. This is easy to realize if \mathbf{y} is sampled at high sampling rate.

To avoid undesirable control chattering a saturation function $\text{sat}(s_i/\phi_i)$ can be used instead of $\text{sgn}(s_i)$ to smooth out the control input inside a boundary layer [4] where ϕ_i may be interpreted as the boundary layer thickness.

4 SIMULATION STUDY

Simplified model of polymerization reactor

A simple two-stage polymerization reactor is illustrated in Fig. 1.

In the simulation study only the first stage of the process was considered. A more detailed model is found in [3]. This was done by using $\mathbf{x} = [m, m_{p1}, T_1]^T$

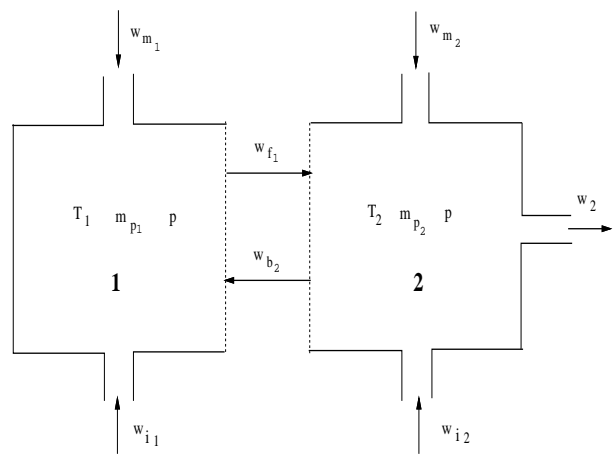


Figure 1: Two-stage polymerization reactor

as the state vector, $\mathbf{u} = [w_{m1}, w_{i1}, w_{f1}]^T$ as the input vector and $\mathbf{y} = [T_1, p]^T$ as the output vector. m is the total mass of the reactor, m_{p1} is the mass of polymer in stage 1, T_1 is the temperature in the first stage, w_{m1} is the monomer feed rate, w_{i1} is the initiator feed rate and w_{f1} is the output rate from stage 1. For this system it is desirable to control the temperature T_1 and pressure p . For simplicity we assume that $u_3 = u_{3s} = \text{const}$. Hence,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned}$$

where

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} u_1 + u_2 \\ a_1 \sqrt{u_2} (x_1 - x_2) - \frac{x_2}{x_1} u_{3s} \\ \frac{1}{x_1} (a_1 a_2 \sqrt{u_2} (x_1 - x_2) + u_1 (a_3 - x_3)) \end{bmatrix}$$

and

$$\mathbf{h}(\mathbf{x}) = [x_3, a_4 x_1 x_3]^T$$

Differentiating \mathbf{y} with respect to time ($\gamma_j = 2$) yields

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{H}_x \mathbf{f} \\ \ddot{\mathbf{y}} &= \frac{\partial}{\partial \mathbf{x}} (\mathbf{H}_x \mathbf{f}) \mathbf{f} + \frac{\partial}{\partial \mathbf{u}} (\mathbf{H}_x \mathbf{f}) \dot{\mathbf{u}} \end{aligned}$$

where \mathbf{H}_x is the Jacobian

$$\mathbf{H}_x = \begin{bmatrix} 0 & 0 & 1 \\ a_4 x_3 & 0 & a_4 x_1 \end{bmatrix}$$

and

$$\frac{\partial}{\partial \mathbf{u}} (\mathbf{H}_x \mathbf{f}) = \begin{bmatrix} \frac{a_3 - x_3}{x_1} & \frac{a_1 a_2 (x_1 - x_2)}{2\sqrt{u_2} x_1} \\ a_3 a_4 & \frac{a_1 a_2 a_4 (x_1 - x_2)}{2\sqrt{u_2}} + a_4 x_3 \end{bmatrix}$$

$$\frac{\partial}{\partial x}(\mathbf{H}_x \mathbf{f}) = \begin{bmatrix} -a_1 a_2 \frac{\sqrt{u_2}}{x_1} (a_1 \sqrt{u_2} (x_1 - x_2) - \frac{u_3 x_2}{x_1}) \\ a_1 a_2 a_4 \sqrt{u_2} (u_1 + u_2) - a_1 a_2 a_4 \sqrt{\frac{u_1}{x_2}} \\ -\frac{u_1}{x_1} (a_1 a_2 \sqrt{u_2} (x_1 - x_2) + u_1 (a_3 - x_3)) \\ (a_1 \sqrt{u_2} (x_1 - x_2) - \frac{u_3 x_2}{x_1}) \\ +\frac{u_1 + u_2}{x_1^2} (-a_3 u_1 + a_1 a_2 \sqrt{u_2} x_2 + u_1 x_3) \\ + a_4 \frac{u_2}{x_1} (a_1 a_2 \sqrt{u_2} (x_1 - x_2) + u_1 (a_3 - x_3)) \end{bmatrix}$$

The constants were chosen as

$$\begin{aligned} a_1 &= 0.04 \sqrt{(kg s)^{-1}} & a_3 &= 300K \\ a_2 &= 286K & a_4 &= 1950 Pa / (kg K) \end{aligned}$$

In the simulation study the nominal model was simply perturbed as

$$\frac{\partial}{\partial \mathbf{x}}(\widehat{\mathbf{H}}_x \mathbf{f}) \hat{\mathbf{f}} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \frac{\partial}{\partial \mathbf{x}}(\mathbf{H}_x \mathbf{f}) \mathbf{f}$$

$$\frac{\partial}{\partial \mathbf{u}}(\widehat{\mathbf{H}}_x \mathbf{f}) = (\mathbf{I} + \mathbf{\Delta}) \frac{\partial}{\partial \mathbf{u}}(\mathbf{H}_x \mathbf{f}), \quad \mathbf{\Delta} = \begin{bmatrix} -0.5 & 0.6 \\ -0.3 & -0.4 \end{bmatrix}$$

The control law was calculated from Eq. 18 as

$$\begin{aligned} \dot{\mathbf{u}} &= \left[\frac{\partial}{\partial \mathbf{u}}(\widehat{\mathbf{H}}_x \mathbf{f}) \right]^{-1} [\dot{\mathbf{y}}_d + \lambda(\mathbf{y}_d - \mathbf{y}) \\ &\quad - \frac{\partial}{\partial \mathbf{x}}(\widehat{\mathbf{H}}_x \mathbf{f}) \hat{\mathbf{f}} - \mathbf{k} \times \text{sgn}(\mathbf{s})] \end{aligned}$$

To avoid chattering the control law was smoothed out in a boundary layer with constant thickness i.e. $\phi = [0.4 \ 0.4]^T$ by simply replacing the signum function with a *sat* (\cdot) function defined as

$$\text{sat}(s_i / \phi_i) = \begin{cases} -1, & \text{if } s_i \leq -\phi_i \\ s_i / \phi_i, & \text{if } -\phi_i < s_i < \phi_i \\ 1, & \text{if } s_i \geq \phi_i \end{cases}$$

The desired output vector was simply chosen as $\dot{\mathbf{y}}_d + \alpha \mathbf{y}_d = \alpha \mathbf{r}$ where α is a positive constant and \mathbf{r} is the commanded input vector. The simulation was performed with two different controllers: (1) a nominal controller (switching gain: $\mathbf{k} = 0$) and (2) a robust sliding controller (switching gain: $\mathbf{k} \neq 0$). The simulation results for a sampling rate of 2 Hz are shown in Figure 2-3. It is seen from the time responses that the sliding controller yields good performance while the tracking for the nominal controller is poor. The sliding controller compensates for model uncertainties, while the nominal case is strongly affected of imprecise cancellations of model nonlinearities. In the nominal case an undesirable chattering in one of the control efforts is also seen. Simulations showed that the difference between the nominal controller and the sliding controller varied according to changes in the operating conditions. In all cases however, we found the sliding mode controller to be the superior one. It should be noted that the steady state error of the nominal controller could be eliminated by including integral action.

Sliding control of MIMO nonlinear *minimum phase* systems has been discussed. Model uncertainties are compensated for by adding a discontinuous term to the controller. Control of a polymerization reactor is used to illustrate the control scheme. Simulation results show that the sliding controller improves the performance significantly when model uncertainties are present.

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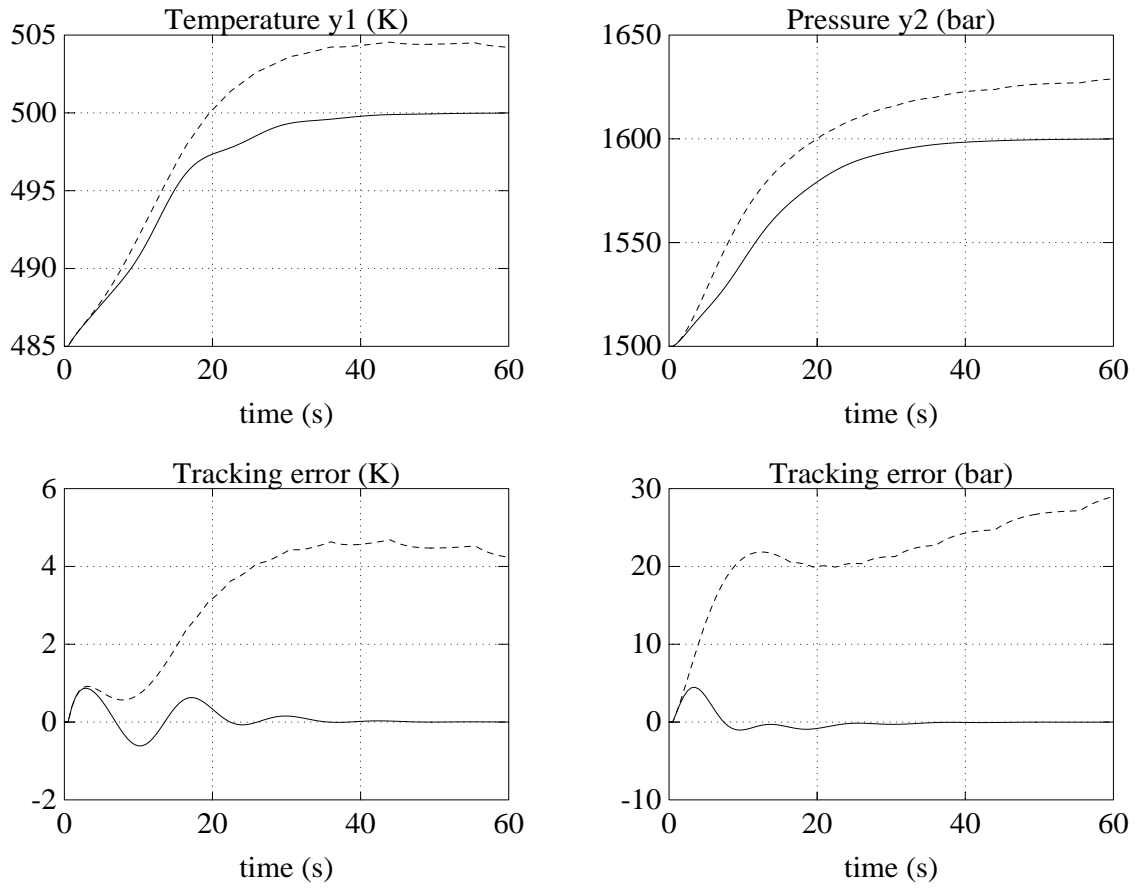


Figure 2: Time responses and tracking errors for the nominal controller (dashed) and the sliding controller (solid) given a commanded input vector: $r = [500(K), 1600(bar)]^T$ for temperature and pressure, respectively.

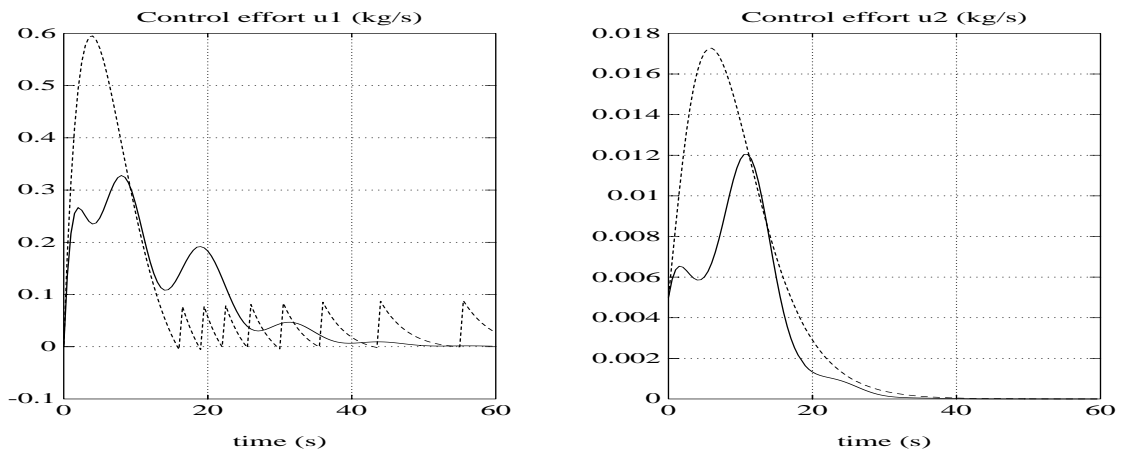


Figure 3: Control inputs for the nominal controller (dashed) and the sliding controller (solid).