Quaternion Feedback Regulation of Underwater Vehicles

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Abstract: Position and attitude set-point regulation of autonomous underwater vehicles (AUVs) in 6 degrees of freedom (DOF) is presented. Euler parameters are used in the representation of global attitude. Three quaternion feedback controllers are derived based on vector quaternion, Euler rotation and Rodrigues parameter feedback, respectively. Global asymptotic stability is proven in the first and second case, while Rodrigues parameter feedback yields asymptotic stability.

Keywords. Position regulation, attitude regulation, quaternion feedback, underwater vehicles.

1 Introduction

Unmanned underwater vehicles (UUV) have become an important tool for undersea operations without divers. Most of them are remotely operated vehicles (ROV) via cable from a surface mother ship. Autonomous underwater vehicles (AUV) are free-swimming devices which carry their own energy source and automatically interact with the environment. Some applications of UUVs and underwater robotic vehicles (URV) are visual survey, inspection of underwater constructions and equipment retriveal. More complex tasks such as underwater welding are also highly actual, especially on aging pipelines and platforms.

For ROVs which are controlled in 6 degrees of freedom (DOF) local autonomy to some extent is required. 6 DOF stationkeeping or tracking of swimming devices are almost impossible tasks for human operators. Also, if the communication channel is narrow-banded, such as an accoustic link, the need for local intelligence is increased. Supervisory control is an important aid for high level teleoperation of both the vehicle and the robot manipulator [11].

For rigid-bodies in 6 DOF the non-linear dynamic equations of motion have a systematic structure which becomes apparent when applying vector notation. This is exploited in the control literature, particularly in the control of mechanical systems like vehicles and robot manipulators. A linear PD control law exploiting the passivity property of robot manipulators was first derived by [9], and later reformulated in [2]. The control law were formulated in both joint-space and in task-space.

For 6 DOF mobile systems the dynamic equations of motion are usually separated into translational and rotational motion. Position is specified by a three vector while various representations of attitude have been discussed in the literature. The most frequently applied representations are the Euler angle conventions, which all are minimal 3-parameter representations. The roll, pitch and yaw (RPY) convention dominates in the context of mobile vehicles. The popularity of Euler angle conventions can probably be explained by their easily understood physical interpretation. However, there are no sensors which can measure the Euler angles directly. Therefore some transformation between the measurement and the parameters must be carried out. Similarly, desired Euler angles must be generated from some desired attitude signal. These properties are shared by all known attitude representations. Hence, nothing is actually gained from knowing the physical interpretation of the Euler angles. There are also some obvious disadvantages in terms of the Euler angle attitude representations. As earlier mentioned, they are 3-parameter representations and therefore they must contain singular points, [8]. The Euler angles are defined by three successive rotations about three axes in a *certain sequence*. This rotation sequence is not exploited in the control design. Applying Euler angles to parameterize rotation matrices $\boldsymbol{R} \in SO(3)$, that is the *Special Orthogonal* group of order 3, implies numerous computations with trigonometric functions. Consequently, it cannot be claimed that Euler angles are better suited than other attitude representations in control applications.

To increase the applicability of an UUV, it should be able to operate at any global attitude. One solution to this is to have two 3-parameter chart representations with singularities at different points. Switching between the charts will then introduce discontinuities in the control law of such systems. This can, however, be avoided by choosing a singularity-free representation such as the *Euler parameters*, see e.g. [4]. Euler parameters, or unit quaternions, have been used in different contexts of attitude control. Control of spacecraft, satellites, aircraft and helicopter are well known applications. More recently the use of Euler parameters has been reported in the robot literature. Quaternion-based attitude set-point regulation has been discussed by [6], [5], [7] and [10] among others. However, the translational motion has not been addressed by these authors. For 6 DOF control problems like underwater vehicles there are significant couplings between the rotational and translational motion. For instance, hydrodynamic added mass will introduce additional couplings due to Coriolis and centrifugal forces. In addition to this, hydrodynamic damping will be strongly coupled. These effects must be considered in the design of a 6 DOF controller.

In this paper we discuss automatic stationkeeping, or dynamic positioning, in 6 DOF for an UUV. The vehicle is shown to have a dynamic model structure similar to standard robot manipulator equations of motion, Section 2. The UUV model is written in terms of Euler parameters to represent attitude. A non-linear 'PD'-control law for position and attitude regulation is presented in Section 3. Three different cases of attitude feedback are discussed.

2 Mathematical Modelling

Kinematic Equations of Motion

The kinematic model describes the geometrical relationship between the earth-fixed and the vehicle-fixed motion. The transformation matrix $\boldsymbol{J}(\boldsymbol{q})$ relates the body-fixed reference frame (B-frame) to the inertial reference frame (I-frame) according to:

$$\dot{\boldsymbol{\xi}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{\nu} \;\Leftrightarrow\; \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}(\boldsymbol{q}) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{4\times3} & \frac{1}{2}\boldsymbol{U}(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \tag{1}$$

where $\boldsymbol{x} = [x\ y\ z]^{\mathrm{T}}$ is the I-frame position of the vehicle, $\boldsymbol{q} = [\eta\ \boldsymbol{\epsilon}^{\mathrm{T}}]^{\mathrm{T}} = [\eta\ \epsilon_1\ \epsilon_2\ \epsilon_3]^{\mathrm{T}}$ is the I-frame unit quaternion representing the attitude, and $\boldsymbol{v} = [u\ v\ w]^{\mathrm{T}}$ and $\boldsymbol{\omega} = [p\ q\ r]^{\mathrm{T}}$ are the linear and angular velocities of the vehicle in the B-frame. The elements of the unit quaternion $\boldsymbol{q} \in SU(2)$, that is the *Special Unitary* group of order 2, are called Euler parameters and they satisfy:

$$\eta^2 + \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{\epsilon} = \eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1 \tag{2}$$

The rotation matrix $\boldsymbol{R}$ from I to B in terms of Euler parameters is written as:

$$\boldsymbol{R}(\boldsymbol{q}) = \begin{bmatrix} \eta^2+\epsilon_1^2-\epsilon_2^2-\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\eta\epsilon_3) & 2(\epsilon_1\epsilon_3+\eta\epsilon_2) \\ 2(\epsilon_1\epsilon_2+\eta\epsilon_3) & \eta^2-\epsilon_1^2+\epsilon_2^2-\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\eta\epsilon_1) \\ 2(\epsilon_1\epsilon_3-\eta\epsilon_2) & 2(\epsilon_2\epsilon_3+\eta\epsilon_1) & \eta^2-\epsilon_1^2-\epsilon_2^2+\epsilon_3^2 \end{bmatrix} \tag{3}$$

The quaternion $\boldsymbol{q}$ can be interpreted as a complex number with η being the real part and $\boldsymbol{\epsilon}$ the complex part. Hence, the complex conjugate of $\boldsymbol{q}$ is defined as:

$$\overline{\boldsymbol{q}} = \begin{bmatrix} \eta \\ -\boldsymbol{\epsilon} \end{bmatrix} \tag{4}$$

Accordingly, the inverse rotation matrix can be written:

$$\boldsymbol{R}^{-1}(\boldsymbol{q}) = \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q}) = \boldsymbol{R}(\overline{\boldsymbol{q}}) \tag{5}$$

Successive rotations involves multiplication between two rotation matrices. It can be shown that:

$$\boldsymbol{R}(\boldsymbol{q}_1)\boldsymbol{R}(\boldsymbol{q}_2) = \boldsymbol{R}(\boldsymbol{q}_1\boldsymbol{q}_2) \tag{6}$$

where quaternion multiplication is defined as ($\boldsymbol{I}_{3\times3}$ is the 3×3 identity matrix):

$$\boldsymbol{q}_1\boldsymbol{q}_2 \triangleq \begin{bmatrix} \eta_1 & -\boldsymbol{\epsilon}_1^{\mathrm{T}} \\ \boldsymbol{\epsilon}_1 & \eta_1\boldsymbol{I}_{3\times3}+\boldsymbol{S}(\boldsymbol{\epsilon}_1) \end{bmatrix} \begin{bmatrix} \eta_2 \\ \boldsymbol{\epsilon}_2 \end{bmatrix} \tag{7}$$

Here we have used the skew-symmetric matrix $\boldsymbol{S}(\boldsymbol{a}) = -\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{a})$ defined as ($\boldsymbol{a} \in \mathbb{R}^3$):

$$\boldsymbol{S}(\boldsymbol{a}) \triangleq \begin{bmatrix} 0 & -a_3 & +a_2 \\ +a_3 & 0 & -a_1 \\ -a_2 & +a_1 & 0 \end{bmatrix} \in SS(3) \tag{8}$$

such that for an arbitrary vector $\boldsymbol{b} \in \mathbb{R}^3$ we have $\boldsymbol{a}\times\boldsymbol{b} \equiv \boldsymbol{S}(\boldsymbol{a})\boldsymbol{b}$. With this notation the coordinate transformation matrix $\boldsymbol{U}(\boldsymbol{q})$ can be written as:

$$\boldsymbol{U}(\boldsymbol{q}) = \begin{bmatrix} -\boldsymbol{\epsilon}^{\mathrm{T}} \\ \eta\boldsymbol{I}_{3\times3}+\boldsymbol{S}(\boldsymbol{\epsilon}) \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\epsilon}^{\mathrm{T}} \\ \boldsymbol{T}(\boldsymbol{q}) \end{bmatrix} \tag{9}$$

Notice that $\boldsymbol{U}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{q} = \boldsymbol{0}$, while $\boldsymbol{T}(\boldsymbol{q})\boldsymbol{\epsilon} = \eta\boldsymbol{\epsilon}$.

Rigid-Body Dynamics

Newton's equations of motion for a rigid-body with respect to the B-frame takes the form:

$$m[\dot{\boldsymbol{v}} + \boldsymbol{\omega}\times\boldsymbol{v} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_G + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_G)] = \boldsymbol{\tau}_1 \tag{10}$$

$$\boldsymbol{I}_0\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\boldsymbol{I}_0\,\boldsymbol{\omega}) + m\boldsymbol{r}_G\times(\dot{\boldsymbol{v}} + \boldsymbol{\omega}\times\boldsymbol{v}) = \boldsymbol{\tau}_2 \tag{11}$$

where $\boldsymbol{r}_G = (x_G, y_G, z_G)^T$ is the center of gravity, m is the constant mass, $\boldsymbol{I}_0$ is the inertia matrix of the vehicle with respect to the B-frame origin, and $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are vectors of external applied forces and moments, respectively. To exploit the structure of the dynamic equations in the control design, we write (10) and (11) in a more compact form as:

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{12}$$

where $\boldsymbol{\tau}_{RB} = [\boldsymbol{\tau}_1^{\mathrm{T}}\ \boldsymbol{\tau}_2^{\mathrm{T}}]^{\mathrm{T}}$ and

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{r}_G) \\ m\boldsymbol{S}(\boldsymbol{r}_G) & \boldsymbol{I}_0 \end{bmatrix} \tag{13}$$

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{v}) + m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{\omega}) \\ -m\boldsymbol{S}(\boldsymbol{v}) + m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{\omega}) & -\boldsymbol{S}(\boldsymbol{I}_0\boldsymbol{\omega}) \end{bmatrix} \tag{14}$$

Notice that the term $m\boldsymbol{S}(\boldsymbol{v})\boldsymbol{v} = \boldsymbol{0}$ is added to make $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ skew-symmetric.

Added Inertia

For a completely submerged vehicle at great depth the hydrodynamic added inertia matrix $\boldsymbol{A}$ is positive definite and constant [1]:

$$\boldsymbol{A} = -\left[\begin{array}{ccc|ccc} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ \hline K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{array}\right] = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \tag{15}$$

where $\boldsymbol{A}_{11} = \boldsymbol{A}_{11}^{\mathrm{T}}$, $\boldsymbol{A}_{12} = \boldsymbol{A}_{21}^{\mathrm{T}}$, and $\boldsymbol{A}_{22} = \boldsymbol{A}_{22}^{\mathrm{T}}$. The concept of added mass introduces Coriolis and sentrifugal terms. These extra terms can be represented by, cf (13) and (14):

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -\boldsymbol{S}(\boldsymbol{A}_{11}\boldsymbol{v} + \boldsymbol{A}_{12}\boldsymbol{\omega}) \\ -\boldsymbol{S}(\boldsymbol{A}_{11}\boldsymbol{v} + \boldsymbol{A}_{12}\boldsymbol{\omega}) & -\boldsymbol{S}(\boldsymbol{A}_{21}\boldsymbol{v} + \boldsymbol{A}_{22}\boldsymbol{\omega}) \end{bmatrix} \tag{16}$$

which is skew-symmetrical.

Hydrodynamic Damping

For an underwater vehicle, the hydrodynamic damping matrix $\boldsymbol{D}(\boldsymbol{\nu})$ should at least include laminar skin friction and viscous damping due to vortex shedding. The matrix $\boldsymbol{D}(\boldsymbol{\nu})$ will be strictly positive, that is:

$$\boldsymbol{D}(\boldsymbol{\nu}) > \boldsymbol{0}_{6\times 6} \tag{17}$$

such that $\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} > 0\ \forall\ \boldsymbol{\nu} \neq \boldsymbol{0}$. This reflects the dissipative nature of the hydrodynamic forces.

Restoring Forces and Moments

The gravitational and buoyant forces, $\boldsymbol{f}_G$ and $\boldsymbol{f}_B$, act through the centre of gravity $\boldsymbol{r}_G = [x_G\ y_G\ z_G]^{\mathrm{T}}$ and the centre of buoyancy $\boldsymbol{r}_B = [x_B\ y_B\ z_B]^{\mathrm{T}}$, respectively. They can be transformed to the B-frame by:

$$\boldsymbol{f}_G = \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q})\begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}, \quad \boldsymbol{f}_B = \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q})\begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} \tag{18}$$

where W and B denote the weight and buoyancy of the underwater vehicle. Notice that the I-frame z-axis is taken to be positive downwards. The restoring forces and moments are collected in the vector $\boldsymbol{g}(\boldsymbol{q})$ according to:

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{f}_B - \boldsymbol{f}_G \\ \boldsymbol{r}_B \times \boldsymbol{f}_B - \boldsymbol{r}_G \times \boldsymbol{f}_G \end{bmatrix} \tag{19}$$

Dynamic Equations of Motion

The rigid-body dynamics together with added inertia, hydrodynamic damping and restoring forces and moments yields the total dynamic model:

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{20}$$

where

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{A} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} \tag{21}$$

$$\boldsymbol{C}(\boldsymbol{\nu}) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}) + \boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -\boldsymbol{S}(\boldsymbol{M}_{11}\boldsymbol{v} + \boldsymbol{M}_{12}\boldsymbol{\omega}) \\ -\boldsymbol{S}(\boldsymbol{M}_{11}\boldsymbol{v} + \boldsymbol{M}_{12}\boldsymbol{\omega}) & -\boldsymbol{S}(\boldsymbol{M}_{22}\boldsymbol{\omega} + \boldsymbol{M}_{21}\boldsymbol{v}) \end{bmatrix} \tag{22}$$

Notice that $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > \boldsymbol{0}_{6\times 6}$ is constant and positive definite, and that $\boldsymbol{C}(\boldsymbol{\nu}) = -\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{\nu})$ is skew-symmetrical. These properties will be exploited in the Lyapunov analysis of the proposed control laws.

Attitude Error Dynamics

The rotation matrix $\boldsymbol{R} \in SO(3)$ from the I-frame to the B-frame represents the actual attitude of the vehicle. Thus, the Euler parameters can be seen as a parameterization of $SO(3)$, that is $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{q})$. Let $\boldsymbol{R}_d$ denote the desired attitude, that is the rotation matrix from the inertial frame to a desired frame (D-frame). The quaternion parameterization is given by

$\boldsymbol{R}_d = \boldsymbol{R}(\boldsymbol{q}_d)$. The control objective is to make the B-frame coincide with the D-frame such that $\boldsymbol{R} = \boldsymbol{R}_d$. The attitude error is defined as $\tilde{\boldsymbol{R}} = \boldsymbol{R}_d^{-1}\boldsymbol{R} = \boldsymbol{R}_d^{\mathrm{T}}\boldsymbol{R}$, and the control objective therefore transforms to $\tilde{\boldsymbol{R}} = \boldsymbol{I}_{6\times 6}$. Applying the Euler parameter representation we obtain $\tilde{\boldsymbol{R}} = \boldsymbol{R}(\tilde{\boldsymbol{q}})$ where:

$$\tilde{\boldsymbol{q}} = \overline{\boldsymbol{q}}_d \boldsymbol{q} = \begin{bmatrix} \eta_d & \boldsymbol{\epsilon}_d^{\mathrm{T}} \\ -\boldsymbol{\epsilon}_d & \eta_d \boldsymbol{I}_{3\times 3} - \boldsymbol{S}(\boldsymbol{\epsilon}_d) \end{bmatrix} \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix} \tag{23}$$

This expression is obtained by combining (5), (6) and (7). Perfect set-point regulation is expressed in quaternion notation as:

$$\boldsymbol{q} = \boldsymbol{q}_d \;\Leftrightarrow\; \tilde{\boldsymbol{q}} = \begin{bmatrix} \pm 1 \\ \boldsymbol{0} \end{bmatrix} \tag{24}$$

The attitude error differential equations follows from (1) and (9), that is:

$$\dot{\tilde{\boldsymbol{q}}} = \frac{1}{2}\boldsymbol{U}(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{\omega}} \quad \Leftrightarrow \quad \begin{cases} \dot{\tilde{\eta}} = -\frac{1}{2}\tilde{\boldsymbol{\epsilon}}^{\mathrm{T}}\tilde{\boldsymbol{\omega}} \\ \dot{\tilde{\boldsymbol{\epsilon}}} = \frac{1}{2}[\tilde{\eta}\boldsymbol{I}_{3\times 3} + \boldsymbol{S}(\tilde{\boldsymbol{\epsilon}})]\tilde{\boldsymbol{\omega}} \end{cases} \tag{25}$$

where the desired angular velocity $\boldsymbol{\omega}_d = \boldsymbol{0}$. Hence, $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d = \boldsymbol{\omega}$. Notice that the attitude error itself has group structure, that is $\tilde{\boldsymbol{q}} \in SU(2)$.

3 Main Results

In this section we propose three different set-point regulators for the AUV model in Section 2. The controllers are formulated in a general framework exploiting the 6 DOF model properties. Attitude is represented by Euler parameters with vector quaternion, Euler rotation and Rodrigues parameter feedback considered in separate sections.

3.1 Vector Quaternion Feedback Control

Feedback from the vector quaternion $\boldsymbol{\epsilon}$ will first be utilized for PD rotational control. Let us define a Lyapunov function as:

$$V_1 = \frac{1}{2}(\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\nu} + \boldsymbol{z}_0^{\mathrm{T}}\boldsymbol{K}_0\boldsymbol{z}_0) \tag{26}$$

where

$$\boldsymbol{z}_0 = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{q}} - \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} \end{bmatrix}, \quad \boldsymbol{K}_0 = \begin{bmatrix} \boldsymbol{K}_x & \boldsymbol{0}_{3\times 4} \\ \boldsymbol{0}_{4\times 3} & c\boldsymbol{I}_{4\times 4} \end{bmatrix} = \boldsymbol{K}_0^{\mathrm{T}} > \boldsymbol{0}_{7\times 7} \tag{27}$$

with $c > 0$ and $\boldsymbol{K}_x = \boldsymbol{K}_x^{\mathrm{T}} > \boldsymbol{0}_{3\times 3}$. Differentiating V_1 with respect to time yields (assuming $\dot{\boldsymbol{x}}_d = \boldsymbol{0}$):

$$\begin{aligned} \dot{V}_1 &= \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{\nu}} + \dot{\boldsymbol{z}}_0^{\mathrm{T}}\boldsymbol{K}_0\boldsymbol{z}_0 \\ &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] + \boldsymbol{\nu}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q}) & \boldsymbol{0}_{3\times 4} \\ \boldsymbol{0}_{3\times 3} & \frac{1}{2}\boldsymbol{U}^{\mathrm{T}}(\tilde{\boldsymbol{q}}) \end{bmatrix} \\ &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] + \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_1 \end{aligned}$$

where

$$\boldsymbol{z}_1 = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{\epsilon}} \end{bmatrix}, \quad \boldsymbol{K}_p(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{K}_x & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \frac{c}{2}\boldsymbol{I}_{3\times 3} \end{bmatrix} \tag{29}$$

Here we have used the fact that $\boldsymbol{M}$ is constant and symmetrical, and that $\boldsymbol{C}(\boldsymbol{\nu})$ is skew-symmetrical. The control law is chosen as:

$$\boldsymbol{\tau} = -\boldsymbol{K}_d\boldsymbol{\nu} - \boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_1 + \boldsymbol{g}(\boldsymbol{q}) \tag{30}$$

whit $\boldsymbol{K}_d = \boldsymbol{K}_d^{\mathrm{T}} > \boldsymbol{0}_{6\times 6}$. This finally yields a negative definite $\dot{V}_1$, that is:

$$\dot{V}_1 = -\boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{K}_d + \boldsymbol{D}(\boldsymbol{\nu})]\boldsymbol{\nu} \le 0 \tag{31}$$

The Lyapunov function time derivative is zero if $\boldsymbol{\nu} = \boldsymbol{0}$. Hence, asymptotic stability cannot be guaranteed by applying Lyapunov's direct method. There are two closed-loop equilibrium points corresponding to $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{0} \;\Rightarrow\; \tilde{\eta} = \pm 1$. Both equilibrium points represent the desired attitude. If $\boldsymbol{q}$ represent one certain attitude, then $-\boldsymbol{q}$ is the same attitude after a $\pm 2\pi$ rotation about an axis. Physically these two points are indistinguishable, but mathematically they are distinct. In fact, $\tilde{\eta} = +1$ is a stable equilibrium point, whereas $\tilde{\eta} = -1$ is unstable. This can be seen from the following discussion. Suppose $\tilde{\eta} = -1$ and $\tilde{\boldsymbol{x}} = \boldsymbol{0}$. The steady-state value of the Lyapunov function is then:

$$V_{1s} = 2c \tag{32}$$

If the system is perturbed to $\tilde{\eta} = -1 + \varepsilon$, $\varepsilon > 0$, it can be shown that V_1 takes the value:

$$V_1 = 2c - \varepsilon < V_{1s} \tag{33}$$

Since V_1 decreases monotonically for $\tilde{\eta} \neq \pm 1$, the system can never return to the unstable equilibrium point. Consequently, application of LaSalle's invariant set theorem, [3], implies (almost) global asymptotic stability. From (31) it is observed that asymptotic stability is obtained even with a simple proportional feedback control law, that is $\boldsymbol{K}_d = \boldsymbol{0}_{6\times 6}$.

The unstable equilibrium point of the vector quaternion feedback control law is awell known attitude control phenomenon. A small perturbation in $\tilde{\boldsymbol{q}}$

will cause the vehicle to rotate an angle of 2π about an axis. This behaviour is closely related to the mathematical properties of the Euler parameters, or unit quaternions in general. The identity on $SU(2)$ is a 4π rotation about an arbitrary axis. Therefore, if attitude is parameterized in terms of unit quaternions, a 2π rotation will be interpreted differently from the identity.

3.2 Euler Rotation Feedback Control

Euler rotation feedback is obtained by substituting the vector quaternion $\boldsymbol{\epsilon}$ with the Euler rotation $2\eta\boldsymbol{\epsilon}$ in the control law. The analysis becomes very similar if the Lyapunov function is chosen according to:

$$V_2 = \frac{1}{2}(\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\nu} + \boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{z}_1) \tag{34}$$

Here $\boldsymbol{z}_1$ are the same as previous, while $\boldsymbol{K}_1$ is defined as:

$$\boldsymbol{K}_1 = \begin{bmatrix} \boldsymbol{K}_x & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & c\boldsymbol{I}_{3\times3} \end{bmatrix} = \boldsymbol{K}_1^{\mathrm{T}} > \boldsymbol{0}_{6\times6} \tag{35}$$

Time differentiation of V_2 gives:

$$\begin{aligned} \dot{V}_2 &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] \\ &+ \boldsymbol{\nu}^{\mathrm{T}}\begin{bmatrix} \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q}) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \frac{1}{2}\boldsymbol{T}^{\mathrm{T}}(\tilde{\boldsymbol{q}}) \end{bmatrix}\boldsymbol{K}_1\boldsymbol{z}_1 \\ &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] + \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_2 \end{aligned} \tag{36}$$

where $\boldsymbol{T}(\tilde{\boldsymbol{q}})$ and $\boldsymbol{K}_p(\boldsymbol{q})$ are defined above and

$$\boldsymbol{z}_2 = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \tilde{\eta}\tilde{\boldsymbol{\epsilon}} \end{bmatrix} \tag{37}$$

Hence, the control law is chosen as:

$$\boldsymbol{\tau} = -\boldsymbol{K}_d\boldsymbol{\nu} - \boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_2 + \boldsymbol{g}(\boldsymbol{q}) \tag{38}$$

and again the Lyapunov function time derivative becomes negative definite, that is:

$$\dot{V}_2 = -\boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{K}_d + \boldsymbol{D}(\boldsymbol{\nu})]\boldsymbol{\nu} \leq 0 \tag{39}$$

This system has three closed-loop equilibrium points, that is $\tilde{\eta} = \pm 1$ and $\tilde{\eta} = 0$. The points $\tilde{\eta} = \pm 1$ both corresponds to the desired attitude. However, in contradiction to the vector quaternion feedback law of Section 3.1, they are both stable. The latter equilibrium point $\tilde{\eta} = 0$ is, however, unstable. Suppose $\tilde{\eta} = 0$ and $\tilde{\boldsymbol{x}} = \boldsymbol{0}$. Hence, the steady-state value of the Lyapunov function becomes:

$$V_{2s} = \frac{c}{2} \tag{40}$$

If the system is perturbed to $\tilde{\eta} = \pm\varepsilon$, V_2 is changed accordingly:

$$V_2 = \frac{c}{2}(1 - \varepsilon^2) < V_{2s} \tag{41}$$

which is similar to the vector quaternion feedback case. Hence, (almost) global asymptotic stability can be stated by applying LaSalle's invariant set theorem.

3.3 Rodrigues Parameter Feedback Control

The last attitude control law is based on Rodrigues parameter. The Rodrigues parameters (Gibb's vector) are defined as:

$$\boldsymbol{\rho} = \boldsymbol{\epsilon}/\eta \tag{42}$$

Notice that $\boldsymbol{\rho}$ is singular for $\eta = 0$. This implies that global asymptotic stability cannot be proven. Let us define the Lyapunov function as:

$$V_3 = \frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\nu} + \frac{1}{2}\tilde{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{K}_x\tilde{\boldsymbol{x}} - c\,\ln|\tilde{\eta}| \tag{43}$$

The time derivative of V_3 is found to be:

$$\begin{aligned} \dot{V}_3 &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] \\ &+ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{K}_x\tilde{\boldsymbol{x}} - \frac{c}{2\tilde{\eta}}\boldsymbol{\omega}^{\mathrm{T}}\tilde{\boldsymbol{\epsilon}} \\ &= \boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{\tau} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{q})] + \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_3 \end{aligned} \tag{44}$$

where

$$\boldsymbol{z}_3 = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{\epsilon}}/\tilde{\eta} \end{bmatrix} \tag{45}$$

This suggests that the control law should be chosen as:

$$\boldsymbol{\tau} = -\boldsymbol{K}_d\boldsymbol{\nu} - \boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_3 + \boldsymbol{g}(\boldsymbol{q}) \tag{46}$$

to yield:

$$\dot{V}_3 = -\boldsymbol{\nu}^{\mathrm{T}}[\boldsymbol{K}_d + \boldsymbol{D}(\boldsymbol{\nu})]\boldsymbol{\nu} \leq 0 \tag{47}$$

The equilibrium points $\tilde{\eta} = \pm 1$ are both stable. However, application of LaSalle's invariant set theorem guarantees only asymptotic stability since we have one singularity at $\eta = 0$.

4 Simulation study

The three controllers proposed in Section 3 were simulated for an underwater vehicle given by the following set of parameters:

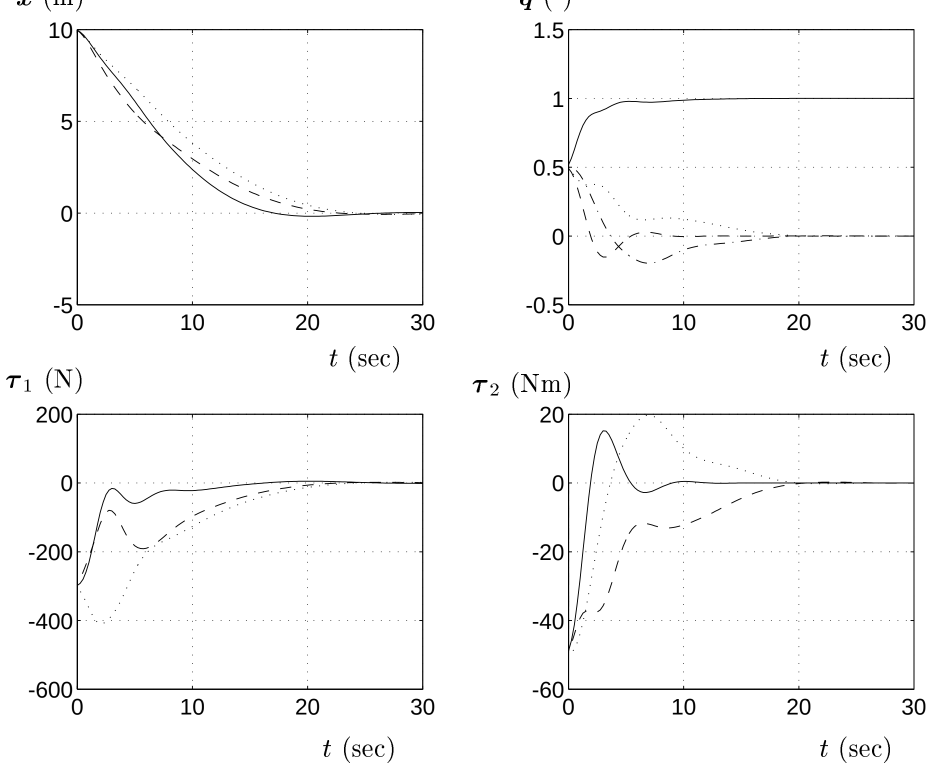

Figure 1: Step response: vector quaternion feedback.

$$\boldsymbol{M} = \text{diag}\{215,\ 265,\ 265,\ 40,\ 80,\ 80\}$$

$$\boldsymbol{D}(\boldsymbol{\nu}) = \text{diag}\{70,\ 100,\ 100,\ 30,\ 50,\ 50\} + \text{diag}\{100|u|,\ 200|v|,\ 200|w|,\ 50|p|,\ 100|q|,\ 100|r|\}$$

The vehicle is assumed to be neutrally buoyant with $W = B = 185 * 9.8$ (N). The control law parameters were set to $\boldsymbol{K}_d = \boldsymbol{I}_{6\times 6}$, $\boldsymbol{K}_x = 30 \cdot \boldsymbol{I}_{3\times 3}$ and $c = 200$ in all three cases. The initial values were $\boldsymbol{\xi}(0) = [10,\ 10,\ 10,\ 0.5,\ 0.5,\ 0.5,\ 0.5]^{\mathrm{T}}$ and $\boldsymbol{\nu}(0) = \mathbf{0}$, and the regulation set-point was chosen as $\boldsymbol{\xi}_d = [0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0]^{\mathrm{T}}$.

We used Runge-Kutta's 4th-order method with sampling time 0.25 (sec) in the simulations. The results are given in Figures 1,2 and 3.

5 Conclusions

Three 6 DOF underwater vehicle control laws for set-point regulation have been dreived. Furthermore, it has been shown that these control laws can be written in a unified framework according to:

$$\boldsymbol{\tau} = -\boldsymbol{K}_d\boldsymbol{\nu} - \boldsymbol{K}_p(\boldsymbol{q})\boldsymbol{z}_i + \boldsymbol{g}(\boldsymbol{q}); \qquad (i = 1, 2, 3) \tag{48}$$

where

Feedback type	State vector
Quaternion	$\boldsymbol{z}_1 = [\tilde{x},\ \tilde{y},\ \tilde{z},\ \tilde{\epsilon_1},\ \tilde{\epsilon_2},\ \tilde{\epsilon_3}]^{\mathrm{T}}$
Euler rotation	$\boldsymbol{z}_2 = [\tilde{x},\ \tilde{y},\ \tilde{z},\ \tilde{\eta}\tilde{\epsilon_1},\ \tilde{\eta}\tilde{\epsilon_2},\ \tilde{\eta}\tilde{\epsilon_3}]^{\mathrm{T}}$
Rodrigues par.	$\boldsymbol{z}_3 = [\tilde{x},\ \tilde{y},\ \tilde{z},\ \tilde{\epsilon_1}/\tilde{\eta},\ \tilde{\epsilon_2}/\tilde{\eta},\ \tilde{\epsilon_3}/\tilde{\eta}]^{\mathrm{T}}$

This simply is a PD-control law with gravitational compensation. Global asymptotic stability is proven in the cases of vector quaternion feedback and Euler rotation feedback while Rodrigues parameter feedback only yields asymptotic stability.

The simulation study indicates that the overall system performance is excellent in all three cases.

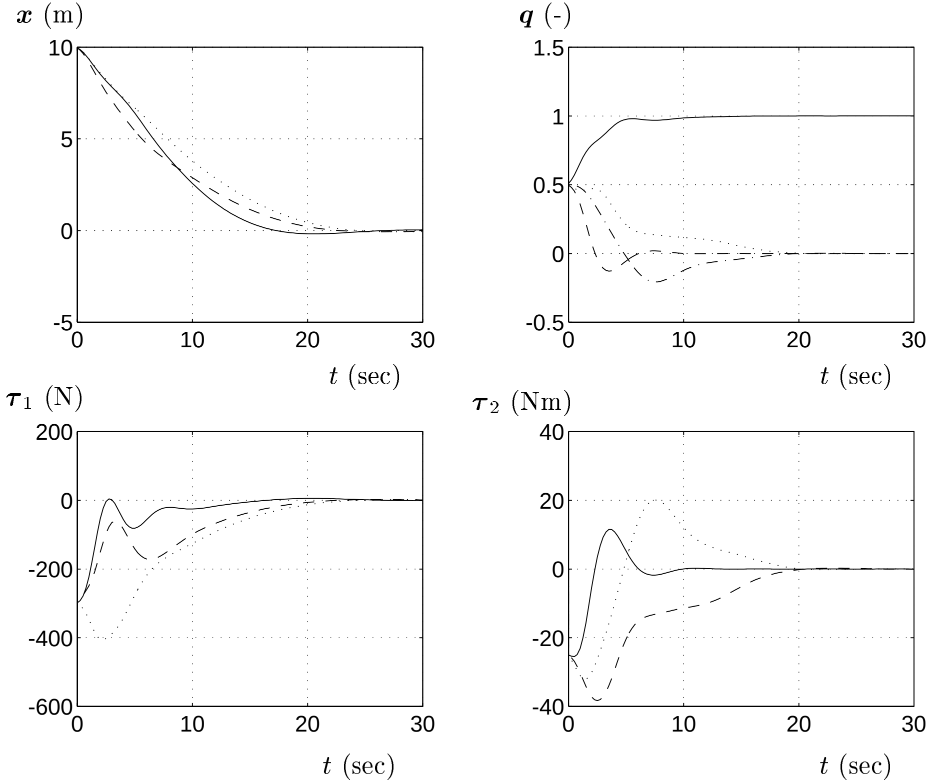

Figure 2: Step response: Euler rotation feedback.

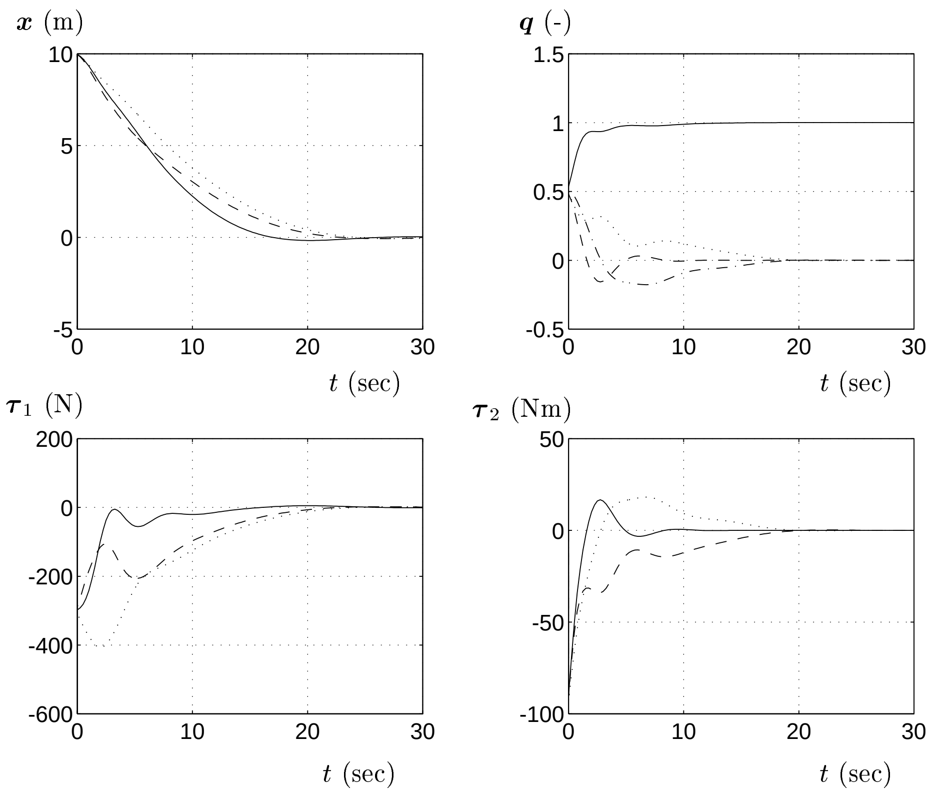

Figure 3: Step response: Rodrigues parameter feedback.

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