Nonlinear Modelling of Marine Vehicles in 6 Degrees of Freedom

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Journal of Mathematical Modelling of Systems, Vol. 1, No. 1, 1995.

Abstract

In this paper a unified framework for vectorial parameterisation of inertia, Coriolis and centrifugal, and hydrodynamic added mass forces for marine vehicles in 6 degrees of freedom (DOF) is presented. Emphasises is placed on representing the 6 DOF nonlinear marine vehicle equations of motion in vector form satisfying certain matrix properties like symmetry, skew-symmetry and positive definiteness. For marine vehicles, this problem was first addressed by Fossen [1]. The proposed representations in this paper are based on extensions of this work. The main results are presented as two theorems. The first theorem is derived by applying Newton's equations whereas the second theorem reviews the main results of Sagatun and Fossen [6] where the Lagrangian formalism is applied.

Key words: Modelling, marine vehicles, equations of motion, Lagrangian and Newtonian mechanics.

1 Introduction

In order to describe the motion of a marine vehicle in 6 *degrees of freedom* (DOF), 6 independent coordinates representing position and attitude are necessary. The 6 different motion components are defined as: *surge, sway, heave, roll, pitch* and *yaw* (see Table 1).

Hence the general motion of a marine vehicle in 6 DOF can be described by the following vectors:

$$\begin{array}{lll}
\boldsymbol{\eta} = [\boldsymbol{\eta}_1^T, \boldsymbol{\eta}_2^T]^T; & \boldsymbol{\eta}_1 = [x, y, z]^T; & \boldsymbol{\eta}_2 = [\phi, \theta, \psi]^T \\
\boldsymbol{\nu} = [\boldsymbol{\nu}_1^T, \boldsymbol{\nu}_2^T]^T; & \boldsymbol{\nu}_1 = [u, v, w]^T; & \boldsymbol{\nu}_2 = [p, q, r]^T \\
\boldsymbol{\tau} = [\boldsymbol{\tau}_1^T, \boldsymbol{\tau}_2^T]^T; & \boldsymbol{\tau}_1 = [X, Y, Z]^T; & \boldsymbol{\tau}_2 = [K, M, N]^T
\end{array}$$

Table 1: 6 DOF motion components for a marine vehicle.

DOF		Forces and Moments	Body-Fixed Velocities	Inertial Position and Euler Angles
1	motions in the x-direction (surge)	X	u	x
2	motions in the y-direction (sway)	Y	v	y
3	motions in the z-direction (heave)	Z	w	z
4	rotation about the x-axis (roll)	K	p	ϕ
5	rotation about the y-axis (pitch)	M	q	θ
6	rotation about the z-axis (yaw)	N	r	ψ

where $\boldsymbol{\eta}$ denotes the earth-fixed position and attitude vector, $\boldsymbol{\nu}$ denotes the body-fixed linear and angular velocity vector and $\boldsymbol{\tau}$ is used to describe the forces and moments acting on the vehicle in the body-fixed frame. The body-fixed and inertial reference frames are shown in Figure 1.

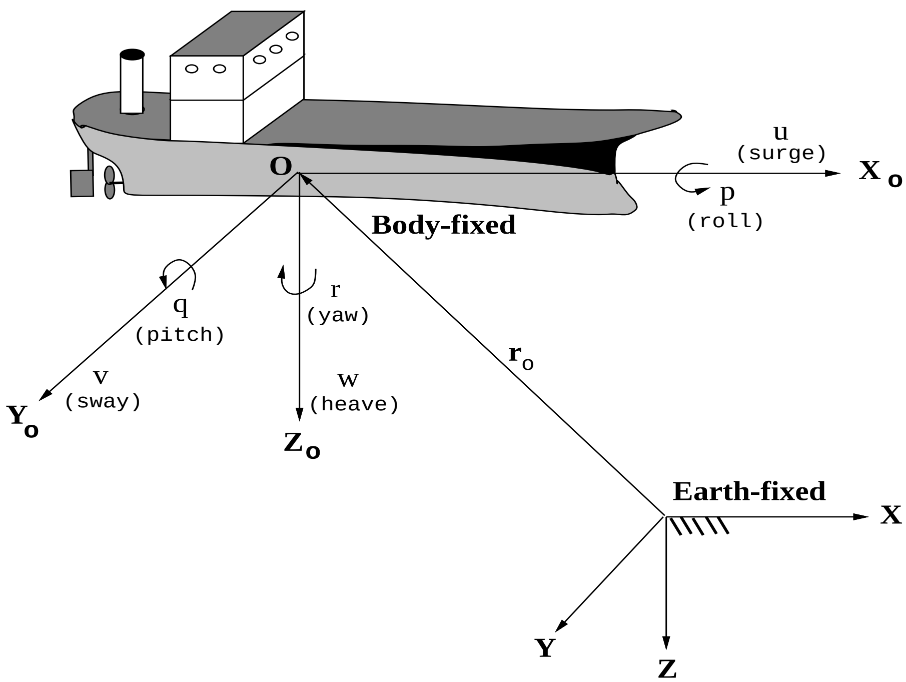

Figure 1: Body-fixed and earth-fixed (inertial) reference frames.

2 Newtonian Approach

The motion of a rigid body with respect to a body-fixed rotating reference frame $X_0Y_0Z_0$ with origin O is given by Newton's laws:

$$m[\dot{\boldsymbol{\nu}}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1 + \dot{\boldsymbol{\nu}}_2 \times \boldsymbol{r}_G + \boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_2 \times \boldsymbol{r}_G)] = \boldsymbol{\tau}_1 \tag{1}$$

$$\boldsymbol{I}_0\, \dot{\boldsymbol{\nu}}_2 + \boldsymbol{\nu}_2 \times (\boldsymbol{I}_0\, \boldsymbol{\nu}_2) + m\boldsymbol{r}_G \times (\dot{\boldsymbol{\nu}}_1 + \boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) = \boldsymbol{\tau}_2 \tag{2}$$

where $\boldsymbol{r}_G = [x_G, y_G, z_G]^T$ is the centre of gravity and $\boldsymbol{I}_0$ is the inertia tensor with respect to the origin O of the body-fixed reference frame, that is:

$$\boldsymbol{I}_0 \triangleq \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}; \quad \boldsymbol{I}_0 = \boldsymbol{I}_0^T > \boldsymbol{0}; \quad \dot{\boldsymbol{I}}_0 = \boldsymbol{0} \tag{3}$$

Here I_x, I_y and I_z are the moments of inertia about the X_0, Y_0 and Z_0 axes and $I_{xy} = I_{yx}, I_{xz} = I_{zx}$ and $I_{yz} = I_{zy}$ are the products of inertia. These equations can be expressed in a more compact form as:

$$\boldsymbol{M}_{RB}\, \dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\, \boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{4}$$

where $\boldsymbol{\nu} = [u, v, w, p, q, r]^T$ is the body-fixed linear and angular velocity vector, $\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^T$ is a generalised vector of external forces and moments and $\boldsymbol{M}_{RB}$ and $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ will be referred to as the rigid-body inertia and, Coriolis and centrifugal matrices, respectively. Before the parameterisation of these matrices are discussed it is convenient to define a skew-symmetric matrix operator.

Definition 1 (Skew-Symmetric Matrix Operator)
Let $\boldsymbol{S}(\cdot)$ be a skew-symmetric matrix operator defined such that the vector cross product $\boldsymbol{a} \times \boldsymbol{b} \triangleq \boldsymbol{S}(\boldsymbol{a})\boldsymbol{b}$, that is:

$$\boldsymbol{S}(\boldsymbol{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \qquad \boldsymbol{S}(\boldsymbol{a}) = -\boldsymbol{S}^T(\boldsymbol{a}) \tag{5}$$

□

2.1 Rigid-Body Inertia Matrix

The rigid-body inertia matrix $\boldsymbol{M}_{RB}$ can be *uniquely* determined from (1) and (2), that is:

$$\boldsymbol{M}_{RB}\, \dot{\boldsymbol{\nu}} \triangleq \begin{bmatrix} m\dot{\boldsymbol{\nu}}_1 + m\dot{\boldsymbol{\nu}}_2 \times \boldsymbol{r}_G \\ \boldsymbol{I}_0\, \dot{\boldsymbol{\nu}}_2 + m\boldsymbol{r}_G \times \dot{\boldsymbol{\nu}}_1 \end{bmatrix} \tag{6}$$

From this expression the positive definite inertia matrix $\boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^T > \boldsymbol{0}$ can be defined according to:

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{r}_G) \\ m\boldsymbol{S}(\boldsymbol{r}_G) & \boldsymbol{I}_0 \end{bmatrix}$$

$$
= \begin{bmatrix} m & 0 & 0 & 0 & mz_G & -my_G \\ 0 & m & 0 & -mz_G & 0 & mx_G \\ 0 & 0 & m & my_G & -mx_G & 0 \\ 0 & -mz_G & my_G & I_x & -I_{xy} & -I_{xz} \\ mz_G & 0 & -mx_G & -I_{yx} & I_y & -I_{yz} \\ -my_G & mx_G & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix} \tag{7}
$$

where $\boldsymbol{I}_{3\times 3}$ is the identity matrix and $\boldsymbol{I}_0$ is the inertia tensor with respect to O.

2.2 Rigid-Body Coriolis and Centripetal Matrix

The Coriolis and centripetal matrix can be represented in numerous ways. Two skew-symmetrical representations motivated from Newton's laws are given by Theorem 1.

Theorem 1 (Parameterisation of Coriolis and Centripetal Forces)
i) *The Coriolis and centripetal matrix can be parameterised according to:*

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_G) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) + m\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_0\, \boldsymbol{\nu}_2) \end{bmatrix} \tag{8}
$$

such that $\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = -\boldsymbol{C}_{RB}^T(\boldsymbol{\nu})$ is skew-symmetrical. Notice that the matrix $\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_G) = [\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{S}(\boldsymbol{\nu}_2)]^T$. Also notice that $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_1 = \boldsymbol{0}$. Equation (8) can be written in component form according to:

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -m(y_G q + z_G r) & m(y_G p + w) & m(z_G p - v) \\ m(x_G q - w) & -m(z_G r + x_G p) & m(z_G q + u) \\ m(x_G r + v) & m(y_G r - u) & -m(x_G p + y_G q) \end{matrix} \right.
$$

$$
\left. \begin{matrix} m(y_G q + z_G r) & -m(x_G q - w) & -m(x_G r + v) \\ -m(y_G p + w) & m(z_G r + x_G p) & -m(y_G r - u) \\ -m(z_G p - v) & -m(z_G q + u) & m(x_G p + y_G q) \\ 0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\ I_{yz} q + I_{xz} p - I_z r & 0 & -I_{xz} r - I_{xy} q + I_x p \\ -I_{yz} r - I_{xy} p + I_y q & I_{xz} r + I_{xy} q - I_x p & 0 \end{matrix} \right] \tag{9}
$$

ii) *An alternative skew-symmetric representation is:*

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_G) - \boldsymbol{S}(\boldsymbol{I}_0\, \boldsymbol{\nu}_2) \end{bmatrix} \tag{10}
$$

which can be written in component form according to:

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & \boldsymbol{C}_{12}(\boldsymbol{\nu}) \\ -\boldsymbol{C}_{12}^T(\boldsymbol{\nu}) & \boldsymbol{C}_{22}(\boldsymbol{\nu}) \end{bmatrix} \tag{11}
$$

where

$$
\boldsymbol{C}_{12}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & m(w-qx_G+py_G) & m(-v-rx_G+pz_G) \\ m(-w+qx_G-py_G) & 0 & m(u-ry_G+qz_G) \\ m(v+rx_G-pz_G) & m(-u+ry_G-qz_G) & 0 \end{bmatrix} \tag{12}
$$

$$
\boldsymbol{C}_{22}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & m(vx_G-uy_G-I_{xz}p-I_{yz}q+I_zr) & m(wx_G-uz_G+I_{xy}p-I_yq+I_{yz}r) \\ m(-vx_G+uy_G+I_{xz}p+I_{yz}q-I_zr) & 0 & m(wy_G-vz_G+I_xp-I_{xy}q-I_{xz}r) \\ m(-wx_G+uz_G-I_{xy}p+I_yq-I_{yz}r) & m(-wy_G+vz_G-I_xp+I_{xy}q+I_{xz}r) & 0 \end{bmatrix} \tag{13}
$$

Proof: *The remaining terms in (1) and (2) after terms associated with* $\dot{\boldsymbol{\nu}}_1$ *and* $\dot{\boldsymbol{\nu}}_2$ *are removed can be written:*

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \triangleq \begin{bmatrix} m[\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_2 \times \boldsymbol{r}_G)] \\ \boldsymbol{\nu}_2 \times (\boldsymbol{I}_0\,\boldsymbol{\nu}_2) + m\boldsymbol{r}_G \times (\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) \end{bmatrix} \tag{14}
$$

By using the formulas:

$$
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{S}(\boldsymbol{a})\boldsymbol{S}(\boldsymbol{b})\boldsymbol{c} \tag{15}
$$

$$
(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{S}(\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b})\boldsymbol{c} \tag{16}
$$

where $\boldsymbol{S}(\boldsymbol{a})\boldsymbol{S}(\boldsymbol{b}) \neq \boldsymbol{S}(\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b})$ *the first line in (14) can be rewritten to obtain the results under i) and ii) in Theorem 1 by noticing that:*

i)

$$
\begin{aligned} & m[\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_2 \times \boldsymbol{r}_G)] \\ &\quad = -m\boldsymbol{\nu}_1 \times \boldsymbol{\nu}_2 - m\boldsymbol{\nu}_2 \times (\boldsymbol{r}_G \times \boldsymbol{\nu}_2) \\ &\quad = -m\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2 - m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{\nu}_2 \end{aligned} \tag{17}
$$

ii)

$$
\begin{aligned} & m[\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_2 \times \boldsymbol{r}_G)] \\ &\quad = -m\boldsymbol{\nu}_1 \times \boldsymbol{\nu}_2 - m(\boldsymbol{\nu}_2 \times \boldsymbol{r}_G) \times \boldsymbol{\nu}_2 \\ &\quad = -m\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2 - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G)\boldsymbol{\nu}_2 \end{aligned} \tag{18}
$$

The second line in (14) can be rewritten in a similar manner. Moreover:

i)

$$
m\boldsymbol{r}_G \times (\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) = m\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1 \tag{19}
$$

ii) *Application of the* Jacobi identity*:*

$$
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0} \tag{20}
$$

to the vector term $\boldsymbol{r}_G \times (\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1)$ *yields:*

$$\boldsymbol{r}_G \times (\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) = -\boldsymbol{\nu}_2 \times (\boldsymbol{\nu}_1 \times \boldsymbol{r}_G) - \boldsymbol{\nu}_1 \times (\boldsymbol{r}_G \times \boldsymbol{\nu}_2) \tag{21}$$

Consequently:

$$\begin{aligned} & m\boldsymbol{r}_G \times (\boldsymbol{\nu}_2 \times \boldsymbol{\nu}_1) \\ & \quad = -m(\boldsymbol{\nu}_2 \times \boldsymbol{r}_G) \times \boldsymbol{\nu}_1 + m(\boldsymbol{\nu}_1 \times \boldsymbol{r}_G) \times \boldsymbol{\nu}_2 \\ & \quad = -m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G)\boldsymbol{\nu}_1 + m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_G)\boldsymbol{\nu}_2 \end{aligned} \tag{22}$$

In addition to this $-m\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_1 = \boldsymbol{0}$ *and* $\boldsymbol{\nu}_2 \times (\boldsymbol{I}_0\,\boldsymbol{\nu}_2) = -\boldsymbol{S}(\boldsymbol{I}_0\boldsymbol{\nu}_2)\boldsymbol{\nu}_2$.

□

Remark to Theorem 1
Two other skew-symmetric representations of $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ *are obtained by omitting the zero term* $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_1 = \boldsymbol{0}$ *and by noticing that* $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2 = -\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1$. *Hence, the results from Theorem 1 can be reformulated according to:*
i)

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_G) \\ m\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_0\,\boldsymbol{\nu}_2) \end{bmatrix} \tag{23}$$

ii)

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) \\ -m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_G) - \boldsymbol{S}(\boldsymbol{I}_0\,\boldsymbol{\nu}_2) \end{bmatrix} \tag{24}$$

3 Lagrangian Approach

In order to include the effects due to hydrodynamic added mass it is advantageous to use a body-fixed energy formulation of *Lagrange's equations* since the vector $\int_0^t \boldsymbol{\nu}\, d\tau$ with $\boldsymbol{\nu} = [u, v, w, p, q, r]^T$ has no immediate physical interpretation. Moreover, *Kirchhoff's equations* which are derived from the ordinary Lagrange equations [5] will be applied for this purpose. Kirchhoff's equations in vector form are written [3]:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_1}\right) + \boldsymbol{\nu}_2 \times \frac{\partial T}{\partial \boldsymbol{\nu}_1} &= \boldsymbol{\tau}_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_2}\right) + \boldsymbol{\nu}_2 \times \frac{\partial T}{\partial \boldsymbol{\nu}_2} + \boldsymbol{\nu}_1 \times \frac{\partial T}{\partial \boldsymbol{\nu}_1} &= \boldsymbol{\tau}_2 \end{aligned} \tag{25}$$

where T is the kinetic energy of the ambient water or the vehicle.

Theorem 2 (Coriolis and Centripetal Matrix from Inertia Matrix)
Let $\boldsymbol{M}$ *be an* 6×6 *inertia matrix defined as:*

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} \tag{26}$$

Hence, the Coriolis and centripetal matrix can always be parameterised according to:

$$\boldsymbol{C}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -\boldsymbol{S}(\boldsymbol{M}_{11}\,\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\,\boldsymbol{\nu}_2) \\ -\boldsymbol{S}(\boldsymbol{M}_{11}\,\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\,\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{M}_{21}\,\boldsymbol{\nu}_1 + \boldsymbol{M}_{22}\,\boldsymbol{\nu}_2) \end{bmatrix} \tag{27}$$

where $\boldsymbol{C}(\boldsymbol{\nu}) = -\boldsymbol{C}^T(\boldsymbol{\nu})$. The results of Theorem 1 can be derived from this expression by letting $\boldsymbol{M} = \boldsymbol{M}_{RB}$.

Proof: *The proof follows Sagatun and Fossen [6] where the kinetic energy T is written as a quadratic form:*

$$T = \frac{1}{2}\boldsymbol{\nu}^T \boldsymbol{M} \boldsymbol{\nu} \tag{28}$$

Expanding this expression yields:

$$T = \frac{1}{2}\left(\boldsymbol{\nu}_1^T \boldsymbol{M}_{11}\boldsymbol{\nu}_1 + \boldsymbol{\nu}_1^T \boldsymbol{M}_{12}\boldsymbol{\nu}_2 + \boldsymbol{\nu}_2^T \boldsymbol{M}_{21}\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2^T \boldsymbol{M}_{22}\boldsymbol{\nu}_2\right) \tag{29}$$

Hence:

$$\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{M}_{11}\,\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\,\boldsymbol{\nu}_2 \tag{30}$$

$$\frac{\partial T}{\partial \boldsymbol{\nu}_2} = \boldsymbol{M}_{21}\,\boldsymbol{\nu}_1 + \boldsymbol{M}_{22}\,\boldsymbol{\nu}_2 \tag{31}$$

From Kirchhoff's equations (25) it is recognised that:

$$\boldsymbol{C}(\boldsymbol{\nu})\,\boldsymbol{\nu} \triangleq \begin{bmatrix} \boldsymbol{\nu}_2 \times \frac{\partial T}{\partial \boldsymbol{\nu}_1} \\ \boldsymbol{\nu}_2 \times \frac{\partial T}{\partial \boldsymbol{\nu}_2} + \boldsymbol{\nu}_1 \times \frac{\partial T}{\partial \boldsymbol{\nu}_1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times 3} & -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_1}) \\ -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_1}) & -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_2}) \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{bmatrix}$$

which after substitution of (30) and (31) proves (27).
□

3.1 Application to the Ambient Water and Vehicle Systems

We will now use Theorem 2 to derive the rigid-body and ambient water Coriolis and centripetal matrices. In order to do this we will define an 6×6 added inertia matrix [4]:

$$
\boldsymbol{M}_A \triangleq - \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix} \tag{32}
$$

The notation of SNAME [7] is used in this expression; for instance the hydrodynamic added mass force Y_A along the y-axis due to an acceleration $\dot{u}$ in the x-direction is written as:

$$
Y_A = Y_{\dot{u}} \dot{u} \quad \text{where} \quad Y_{\dot{u}} \triangleq \frac{\partial Y}{\partial \dot{u}} \tag{33}
$$

For control applications we can assume that $\boldsymbol{M}_A > \mathbf{0}$ is constant and strictly positive. This is based on the assumption that $\boldsymbol{M}_A$ is independent of the wave frequency which is a good approximation for low-frequency control applications. For certain control applications like operation of underwater vehicles at great depth and positioning of surface ships we can also assume that $\boldsymbol{M}_A$ is symmetrical. However, this is not true for a surface ship moving at some speed [2]. For notational simplicity we will write:

$$
\boldsymbol{M}_A = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \qquad \boldsymbol{M}_{RB} = \begin{bmatrix} m\, \boldsymbol{I}_{3\times 3} & -m\, \boldsymbol{S}(\boldsymbol{r}_G) \\ m\, \boldsymbol{S}(\boldsymbol{r}_G) & \boldsymbol{I}_0 \end{bmatrix} \tag{34}
$$

for hydrodynamic added inertia and rigid-body inertia, respectively. Notice that the hydrodynamic added inertia and rigid-body inertia matrices are the same both in the Newtonian and Lagrangian approaches.

Rigid-Body Coriolis and Centripetal Matrix

Theorem 2 with $\boldsymbol{M}_{RB}$ defined in (34) can now be used to derive the following expression for the rigid-body Coriolis and centripetal matrix:

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_G) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_G) - \boldsymbol{S}(\boldsymbol{I}_0\, \boldsymbol{\nu}_2) \end{bmatrix} \tag{35}
$$

This is equivalent to case ii) of Theorem 1. Case i) can also be derived from this expression.

Hydrodynamic Added Coriolis and Centripetal Matrix

In addition to the rigid-body Coriolis and centripetal matrix, an expression for the hydrodynamic added Coriolis and centripetal matrix can be derived. Moreover, Theorem 2 with $\boldsymbol{M}_A$ defined in (34) yields:

$$
\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -\boldsymbol{S}(\boldsymbol{A}_{11}\,\boldsymbol{\nu}_1 + \boldsymbol{A}_{12}\,\boldsymbol{\nu}_2) \\ -\boldsymbol{S}(\boldsymbol{A}_{11}\,\boldsymbol{\nu}_1 + \boldsymbol{A}_{12}\,\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{A}_{21}\,\boldsymbol{\nu}_1 + \boldsymbol{A}_{22}\,\boldsymbol{\nu}_2) \end{bmatrix} \tag{36}
$$

which can be written in component form according to:

$$
\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix} \tag{37}
$$

where

$$
\begin{aligned}
a_1 &= X_{\dot{u}} u + X_{\dot{v}} v + X_{\dot{w}} w + X_{\dot{p}} p + X_{\dot{q}} q + X_{\dot{r}} r \\
a_2 &= X_{\dot{v}} u + Y_{\dot{v}} v + Y_{\dot{w}} w + Y_{\dot{p}} p + Y_{\dot{q}} q + Y_{\dot{r}} r \\
a_3 &= X_{\dot{w}} u + Y_{\dot{w}} v + Z_{\dot{w}} w + Z_{\dot{p}} p + Z_{\dot{q}} q + Z_{\dot{r}} r \\
b_1 &= X_{\dot{p}} u + Y_{\dot{p}} v + Z_{\dot{p}} w + K_{\dot{p}} p + K_{\dot{q}} q + K_{\dot{r}} r \\
b_2 &= X_{\dot{q}} u + Y_{\dot{q}} v + Z_{\dot{q}} w + K_{\dot{q}} p + M_{\dot{q}} q + M_{\dot{r}} r \\
b_3 &= X_{\dot{r}} u + Y_{\dot{r}} v + Z_{\dot{r}} w + K_{\dot{r}} p + M_{\dot{r}} q + N_{\dot{r}} r
\end{aligned} \tag{38}
$$

An attractive simplification of this expression is to assume that:

$$
\boldsymbol{M}_A = -\mathrm{diag}\{\, X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}, K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}} \,\} \tag{39}
$$

$$
\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}} w & Y_{\dot{v}} v \\ 0 & 0 & 0 & Z_{\dot{w}} w & 0 & -X_{\dot{u}} u \\ 0 & 0 & 0 & -Y_{\dot{v}} v & X_{\dot{u}} u & 0 \\ 0 & -Z_{\dot{w}} w & Y_{\dot{v}} v & 0 & -N_{\dot{r}} r & M_{\dot{q}} q \\ Z_{\dot{w}} w & 0 & -X_{\dot{u}} u & N_{\dot{r}} r & 0 & -K_{\dot{p}} p \\ -Y_{\dot{v}} v & X_{\dot{u}} u & 0 & -M_{\dot{q}} q & K_{\dot{p}} p & 0 \end{bmatrix} \tag{40}
$$

4 Equations of Motion

Assume that the external forces and moments $\boldsymbol{\tau}_{RB}$ in (4) is written:

$$
\boldsymbol{\tau}_{RB} = -\underbrace{\boldsymbol{M}_A\,\dot{\boldsymbol{\nu}} - \boldsymbol{C}_A(\boldsymbol{\nu})\,\boldsymbol{\nu}}_{\text{added mass}} - \underbrace{\boldsymbol{D}(\boldsymbol{\nu})\,\boldsymbol{\nu}}_{\text{damping}} - \underbrace{\boldsymbol{g}(\boldsymbol{\eta})}_{\text{restoring forces}} + \underbrace{\boldsymbol{\tau}}_{\text{control input}} \tag{41}
$$

Hence, the marine vehicle equations of motion can be expressed according to:

$$
\boxed{\boldsymbol{M}\,\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\,\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\,\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{\tau}} \tag{42}
$$

where

$$
\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A; \qquad \boldsymbol{C}(\boldsymbol{\nu}) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}) + \boldsymbol{C}_A(\boldsymbol{\nu}) \tag{43}
$$

The matrix $\boldsymbol{D}(\boldsymbol{\nu})$ can be interpreted as a non-symmetrical hydrodynamic damping matrix while $\boldsymbol{g}(\boldsymbol{\eta})$ is included to describe buoyant and gravitational forces. Notice that these equations satisfy:

$$\dot{\boldsymbol{M}} = \mathbf{0}; \quad \boldsymbol{M} = \boldsymbol{M}^T > \mathbf{0}; \quad \boldsymbol{C}(\boldsymbol{\nu}) = -\boldsymbol{C}^T(\boldsymbol{\nu}); \quad \boldsymbol{D}(\boldsymbol{\nu}) > \mathbf{0} \tag{44}$$

For robot manipulators it is common to transform the desired state trajectory to joint space coordinates by applying the inverse kinematics. Hence the kinematic transformation matrix can be avoided in the control scheme. This is not possible for a marine vehicle since $\int_0^t \boldsymbol{\nu}\, d\tau$ has no physical interpretation. Hence, the kinematic equations of motion must be included in the control design if position and attitude control are of interest. For this purpose it is common to apply an Euler angle representation (xyz-convention):

$$\boxed{\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\,\boldsymbol{\nu}} \tag{45}$$

where $\boldsymbol{J}(\boldsymbol{\eta}) = \mathrm{diag}\{\boldsymbol{J}_1(\boldsymbol{\eta}), \boldsymbol{J}_2(\boldsymbol{\eta})\}$ and:

$$\boldsymbol{J}_1(\boldsymbol{\eta}) = \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\theta & -\mathrm{s}\psi\mathrm{c}\phi + \mathrm{c}\psi\mathrm{s}\theta\mathrm{s}\phi & \mathrm{s}\psi\mathrm{s}\phi + \mathrm{c}\psi\mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\psi\mathrm{c}\theta & \mathrm{c}\psi\mathrm{c}\phi + \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi & -\mathrm{c}\psi\mathrm{s}\phi + \mathrm{s}\theta\mathrm{s}\psi\mathrm{c}\phi \\ -\mathrm{s}\theta & \mathrm{c}\theta\mathrm{s}\phi & \mathrm{c}\theta\mathrm{c}\phi \end{bmatrix} \tag{46}$$

$$\boldsymbol{J}_2(\boldsymbol{\eta}) = \begin{bmatrix} 1 & \mathrm{s}\phi\mathrm{t}\theta & \mathrm{c}\phi\mathrm{t}\theta \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & \mathrm{s}\phi/\mathrm{c}\theta & \mathrm{c}\phi/\mathrm{c}\theta \end{bmatrix} \tag{47}$$

This representation is undefined for a pitch angle of $\pm 90^o$ (deg). The above system satisfies:

$$\boldsymbol{\nu}^T\boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} = 0; \qquad \dot{\boldsymbol{\eta}}^T[\dot{\boldsymbol{M}}_\eta - 2\boldsymbol{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})]\,\dot{\boldsymbol{\eta}} = 0 \tag{48}$$

where

$$\boldsymbol{M}_\eta(\boldsymbol{\eta}) = \boldsymbol{J}^{-T}\boldsymbol{M}\boldsymbol{J}^{-1}; \qquad \boldsymbol{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) = \boldsymbol{J}^{-T}(\boldsymbol{C} - \boldsymbol{M}\boldsymbol{J}^{-1}\dot{\boldsymbol{J}})\boldsymbol{J}^{-1} \tag{49}$$

which implies that the theory of passive adaptive control can be applied to design a stable control system.

5 Conclusions

In this paper methods for parameterisation of nonlinear Coriolis, centripetal and inertia forces for marine vehicles in terms of matrix properties like symmetry, skew-symmetry and positiveness have been discussed. Emphasises have been placed on deriving a model of the ambient water of the vehicle by considering the effects of hydrodynamic added inertia. The resulting model is written in vector form. Marine vehicle control design based on Lyapunov stability analyses for the proposed model structure is discussed in [2] and references therein.

6 References

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