Nonlinear Vectorial Backstepping Design for Global Exponential Tracking of Marine Vessels in the Presence of Actuator Dynamics

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Abstract

A nonlinear vectorial backstepping control law for commercial ships is derived by using the concept of vectorial backstepping. Vectorial backstepping is done in 3 steps corresponding to the state vectors of the ship dynamics, kinematics and actuator dynamics. Emphasizes is placed on compensation of the actuator dynamics since the bandwidth of the propellers, thrusters and rudders often is close to the bandwidth of the ship dynamics. Global exponential tracking is proven by applying Lyapunov stability analysis. The case study is simultaneously global exponential tracking of the surge and sway positions (x, y) and the yaw angle ψ of a surface ship. This can only be done by applying nonlinear control theory due to the nonlinear structure of the kinematic equations, Coriolis and centripetal forces, and hydrodynamic damping forces.

1 Introduction

Conventional ship control systems are designed under the assumption that the kinematic and dynamic equations of motion can be linearized such that gain-scheduling techniques and optimal control theory can be applied, see Fossen and Grøvlen [4]. This is not a good assumption for tracking applications where the surge and sway positions (x, y) and yaw angle ψ must be controlled simultaneously. The main reason for this, is that the rotation matrix in yaw, typically must be linearized about 36 operating points (steps of 10 degrees) to cover the whole circle arc with adequate accuracy. In addition to this, assumptions like linear damping and negligible Coriolis and centripetal forces are only good for low-speed applications, that is station-keeping and dynamic positioning (DP). These limitations clearly motivate a nonlinear design.

Control design for marine vessels is usually done under the assumption that the actuator dynamics can be neglected e.g. by choosing the bandwidth of the control law sufficient low. This is quite restrictive since actuators like propellers, thrusters and rudders have time constants in the range of 1–5 (s) which is close to the bandwidth of most ships, i.e. 0.01–0.1 (rad/s) [1]. In this paper the concept of *vectorial backstepping* is applied, see Fossen and Grøvlen [4] and Krstic et al. [5], since this allows the designer to incorporate the effects of the actuator dynamics in a systematic manner. The term *vectorial backstepping* is used since the control law is derived in a vectorial setting by using only 3 steps corresponding to the state vectors of the ship dynamics, kinematics and actuator dynamics. Global exponential stability (GES) of the tracking control law is proven by applying Lyapunov stability analysis. This is done by exploiting nonlinear system properties like symmetry of the inertia matrix, dissipative damping and skew-symmetry of the Coriolis and centripetal matrix, see Fossen [1], [2] and Fossen and Fjellstad [3]. The proposed control law is simulated on a supply vessel.

2 The Idea of Vectorial Backstepping

In Section 3, the dynamics of a marine vessel is written in a vectorial setting motivated by the notation and structural form used in the robot literature. Before the marine vehicle control problem is addressed, we will first demonstrate the idea of *vectorial backstepping* by considering a robot manipulator. Moreover, consider the nonlinear robot model [6]:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v} && (1) \\ \mathbf{M}(\mathbf{q})\dot{\mathbf{v}} + \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v} + \mathbf{g}(\mathbf{q}) &= \boldsymbol{\tau} && (2) \end{aligned}$$

where $\mathbf{M}(\mathbf{q}) = \mathbf{M}^T(\mathbf{q}) > \mathbf{0}$ is the inertia matrix, $\mathbf{C}(\mathbf{q},\mathbf{v})$ is a matrix of Coriolis and centripetal terms defined in terms of the *Christoffel symbols* and $\mathbf{g}(\mathbf{q})$ is a vector of gravitational forces and moments. $\mathbf{q} \in \mathbf{R}^n$ is a vector of joint angles, $\mathbf{v} \in \mathbf{R}^n$ is a vector of joint angular rates and $\boldsymbol{\tau} \in \mathbf{R}^n$ is a vector of control torques. The main idea of vectorial backstepping is illustrated in Example 1.

Example 1 (Vectorial Backstepping) *Vectorial backstepping of a robot manipulator can be done in two steps:*

Step 1:

Define the virtual control*:*

$$\dot{\mathbf{q}} = \mathbf{v} \triangleq \mathbf{s} + \boldsymbol{\alpha} \qquad (3)$$

where $\mathbf{s}$ *is a new state variable and* $\boldsymbol{\alpha}$ *is stabilizing function which can be chosen as:*

$$\boldsymbol{\alpha} = \mathbf{v}_r, \quad \mathbf{v}_r = \boldsymbol{\nu}_d - \boldsymbol{\Lambda}\bar{\mathbf{q}} \qquad (4)$$

where $\boldsymbol{\Lambda} > \mathbf{0}$ *is a diagonal design matrix and* $\bar{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$ *is the tracking error. Combining (3) and (4) yields:*

$$\bar{\mathbf{v}} = -\boldsymbol{\Lambda}\bar{\mathbf{q}} + \mathbf{s} \qquad (5)$$

where $\dot{\bar{\mathbf{q}}} = \bar{\mathbf{v}}$.

Step 2:

Consider the Lyapunov function candidate:

$$V = \frac{1}{2}(\mathbf{s}^T\mathbf{M}(\mathbf{q})\mathbf{s} + \bar{\mathbf{q}}^T\mathbf{K}_q\bar{\mathbf{q}}) > 0 \quad \forall \mathbf{s} \neq \mathbf{0}, \bar{\mathbf{q}} \neq \mathbf{0} \qquad (6)$$

$$\begin{aligned} \dot{V} &= \mathbf{s}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{s}} + \frac{1}{2}\mathbf{s}^T\dot{\mathbf{M}}(\mathbf{q})\mathbf{s} + \bar{\mathbf{q}}^T\mathbf{K}_q\bar{\mathbf{v}} \\ &= \mathbf{s}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{s}} + \frac{1}{2}\mathbf{s}^T\dot{\mathbf{M}}(\mathbf{q})\mathbf{s} \\ &\quad -\bar{\mathbf{q}}^T\mathbf{K}_q\boldsymbol{\Lambda}\bar{\mathbf{q}} + \bar{\mathbf{q}}^T\mathbf{K}_q\mathbf{s} \qquad (7) \end{aligned}$$

Eqns. (3) and (4) can be combined to give:

$$\begin{aligned} \mathbf{M}(\mathbf{q})\dot{\mathbf{s}} &= \mathbf{M}(\mathbf{q})\dot{\mathbf{v}} - \mathbf{M}(\mathbf{q})\dot{\boldsymbol{\alpha}} \\ &= \boldsymbol{\tau} - \mathbf{M}(\mathbf{q})\dot{\mathbf{v}}_r - \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v}_r - \mathbf{g}(\mathbf{q}) \\ &\quad -\mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{s} \qquad (8) \end{aligned}$$

Substituting (8) into (7) yields:

$$\begin{aligned} \dot{V} &= \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{M}(\mathbf{q})\dot{\mathbf{v}}_r - \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v}_r - \mathbf{g}(\mathbf{q}) + \mathbf{K}_q\bar{\mathbf{q}}) \\ &\quad + \mathbf{s}^T(\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q},\mathbf{v}))\mathbf{s} - \bar{\mathbf{q}}^T\mathbf{K}_q\boldsymbol{\Lambda}\bar{\mathbf{q}} \\ &= \mathbf{s}^T(\boldsymbol{\tau} - \mathbf{M}(\mathbf{q})\dot{\mathbf{v}}_r - \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v}_r - \mathbf{g}(\mathbf{q}) \\ &\quad + \mathbf{K}_q\bar{\mathbf{q}}) - \bar{\mathbf{q}}^T\mathbf{K}_q\boldsymbol{\Lambda}\bar{\mathbf{q}} \qquad (9) \end{aligned}$$

Here we have used the skew-symmetric property $\mathbf{s}^T(\frac{1}{2}\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q},\mathbf{v}))\mathbf{s} = \mathbf{0}, \forall \mathbf{s}$. *This suggests that the control law can be chosen as:*

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\dot{\mathbf{v}}_r + \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v}_r + \mathbf{g}(\mathbf{q}) - \mathbf{K}_d\mathbf{s} - \mathbf{K}_q\bar{\mathbf{q}} \qquad (10)$$

where $\mathbf{K}_d = \mathbf{K}_d^T > \mathbf{0}$ *and* $\mathbf{K}_q = \mathbf{K}_q^T > \mathbf{0}$ *are design matrices. This finally yields:*

$$\dot{V} = -\mathbf{s}^T\mathbf{K}_d\mathbf{s} - \bar{\mathbf{q}}^T\mathbf{K}_q\boldsymbol{\Lambda}\bar{\mathbf{q}} < 0 \quad \forall \mathbf{s} \neq \mathbf{0}, \bar{\mathbf{q}} \neq \mathbf{0} \qquad (11)$$

Remark 1 *The control law (10) is GES since:*

$$V(t) < e^{-2\alpha t}V(0) \qquad (12)$$

$$\alpha = \frac{\min(\lambda_{\min}\{\mathbf{K}_d\}, \lambda_{\min}\{\mathbf{K}_q\boldsymbol{\Lambda}\})}{\max(M_M, \lambda_{\max}\{\mathbf{K}_q\})} > 0 \qquad (13)$$

where M_M *is the upper bound on the inertia matrix satisfying:*

$$0 < M_m < \|\mathbf{M}(\mathbf{q})\| < M_M, \quad \forall \mathbf{q} \qquad (14)$$

Remark 2 *The control law (10) is equivalent with the control law of Slotine and Lie [7] with perfectly known parameters (non-adaptive case) except for the additional feedback term* $\mathbf{K}_q\bar{\mathbf{q}}$ *which is necessary to obtain GES and* $\boldsymbol{\Lambda}$ *which replaces the scalar weight* λ.

3 Nonlinear Control of Marine Vessels

It is possible to generalize the nonlinear robot controller in Example 1 to surface ships described by the following model class, Fossen [1], [2]:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} && (15) \\ \mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) &= \mathbf{B}\mathbf{u} && (16) \\ \mathbf{T}\dot{\mathbf{u}} + \mathbf{u} &= \mathbf{u}_c && (17) \end{aligned}$$

$\mathbf{M}$	$\mathcal{R}^{n\times n}$	Inertia matirx including hydrodynamic added inertia
$\mathbf{C}(\boldsymbol{\nu})$	$\mathcal{R}^{n\times n}$	Coriolis/sentripetal matrix incl. hydrodyn. added inertia
$\mathbf{D}(\boldsymbol{\nu})$	$\mathcal{R}^{n\times n}$	Hydrodynamic damp. matrix (skin friction, wave drift, vortex shedding, potential damp.)
$\mathbf{g}(\boldsymbol{\eta})$	$\mathcal{R}^{n\times 1}$	Grav. and buoyancy forces
$\mathbf{B}$	$\mathcal{R}^{n\times r}$	Input matrix (configuration of thrusters and propellers)
$\mathbf{T}$	$\mathcal{R}^{r\times r}$	Diagonal matrix of actuator time constants
$\mathbf{J}(\boldsymbol{\eta})$	$\mathcal{R}^{n\times n}$	Rotation matrix in yaw
$\boldsymbol{\nu}$	$\mathcal{R}^{n\times 1}$	Body-fixed velocity vector
$\boldsymbol{\eta}$	$\mathcal{R}^{n\times 1}$	Position/attitude vector
$\mathbf{u}$	$\mathcal{R}^{r\times 1}$	Vector of actuator states
$\mathbf{u}_c$	$\mathcal{R}^{r\times 1}$	Vector of control inputs

Table 1: Marine vessel notation.

This model describes the motion of a surface ship in *n=3* degrees of freedom (DOF). It is assumed that $r \geq n$ control inputs are available.

Let p.d. denote *positive definite*, s.p. *strictly positive* and s.s. *skew-symmetrical* matrices.

Assumption 1: *The system (15), (16) and (17) satisfies the following properties:*

- *(i)* $\mathbf{M} = \mathbf{M}^T$ is p. d. $\Rightarrow \mathbf{x}^T\mathbf{M}\mathbf{x} > 0, \forall \mathbf{x} \neq \mathbf{0}$
- *(ii)* $\mathbf{C}(\boldsymbol{\nu}) = -\mathbf{C}^T(\boldsymbol{\nu})$ is s.s. $\Rightarrow \mathbf{x}^T\mathbf{C}(\boldsymbol{\nu})\mathbf{x} = \mathbf{0}, \forall \mathbf{x}$
- *(iii)* $\mathbf{D}(\boldsymbol{\nu})$ is s. p. $\Rightarrow \mathbf{x}^T\mathbf{D}(\boldsymbol{\nu})\mathbf{x} = \frac{1}{2}\mathbf{x}^T(\mathbf{D}(\boldsymbol{\nu}) + \mathbf{D}^T(\boldsymbol{\nu}))\mathbf{x} > 0, \forall \mathbf{x} \neq \mathbf{0}$
- *(iv)* $\mathbf{B}\mathbf{B}^T$ and $\mathbf{T}$ are non-singular
- *(v)* $\mathbf{J}(\boldsymbol{\eta})$ is the rotation matrix in yaw $\Rightarrow \mathbf{J}^{-1}(\boldsymbol{\eta}) = \mathbf{J}^T(\boldsymbol{\eta}), \forall \boldsymbol{\eta}$

Assumption 2: *The reference trajectory* $\boldsymbol{\eta}_d^{(3)}$, $\ddot{\boldsymbol{\eta}}_d, \dot{\boldsymbol{\eta}}_d$ *and* $\boldsymbol{\eta}_d$ *is smooth and bounded.*

Definition 1: *The virtual reference trajectories in body-fixed and earth-fixed coordinates are defined as:*

$$\dot{\boldsymbol{\eta}}_r \triangleq \dot{\boldsymbol{\eta}}_d - \boldsymbol{\Lambda}\bar{\boldsymbol{\eta}} \tag{18}$$

$$\boldsymbol{\nu}_r \triangleq \mathbf{J}^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}_r \tag{19}$$

where $\bar{\boldsymbol{\eta}} = \boldsymbol{\eta} - \boldsymbol{\eta}_d$ *is the earth-fixed tracking error and* $\boldsymbol{\Lambda} > \mathbf{0}$ *is a diagonal design matrix.*

Definition 2: *The measure of tracking is defined as:*

$$\mathbf{s} \triangleq \dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}_r = \dot{\bar{\boldsymbol{\eta}}} + \boldsymbol{\Lambda}\bar{\boldsymbol{\eta}} \tag{20}$$

The marine vehicle dynamics (15) and (16) can be written, Fossen [1]:

$$\begin{aligned}\mathbf{M}_\eta(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}} + \mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\dot{\boldsymbol{\eta}} + \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\dot{\boldsymbol{\eta}} + \mathbf{g}_\eta(\boldsymbol{\eta}) \\ = \mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{B}\mathbf{u}\end{aligned} \tag{21}$$

where:

$$\begin{aligned}\mathbf{M}_\eta(\boldsymbol{\eta}) &= \mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{M}\mathbf{J}^{-1}(\boldsymbol{\eta}) \\ \mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) &= \mathbf{J}^{-T}(\boldsymbol{\eta})[\mathbf{C}(\boldsymbol{\nu}) \\ &\quad -\mathbf{M}\mathbf{J}^{-1}(\boldsymbol{\eta})\dot{\mathbf{J}}(\boldsymbol{\eta})]\mathbf{J}^{-1}(\boldsymbol{\eta}) \\ \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) &= \mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{D}(\boldsymbol{\nu})\mathbf{J}^{-1}(\boldsymbol{\eta}) \\ \mathbf{g}_\eta(\boldsymbol{\eta}) &= \mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{g}(\boldsymbol{\eta})\end{aligned}$$

Definitions 1 and 2 can be used to write the marine vehicle dynamics in the following form:

$$\begin{aligned}\mathbf{M}_\eta(\boldsymbol{\eta})\dot{\mathbf{s}} &= -\mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} - \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} \\ &\quad +\mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{B}\mathbf{u} - \mathbf{M}_\eta(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}}_r - \mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\dot{\boldsymbol{\eta}}_r \\ &\quad -\mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\dot{\boldsymbol{\eta}}_r - \mathbf{g}_\eta(\boldsymbol{\eta})\end{aligned} \tag{22}$$

or equivalently:

$$\begin{aligned}\mathbf{M}_\eta(\boldsymbol{\eta})\dot{\mathbf{s}} &= -\mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} - \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} \\ &\quad +\mathbf{J}^{-T}(\boldsymbol{\eta})[\mathbf{B}\mathbf{u} - \mathbf{M}\dot{\boldsymbol{\nu}}_r - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r \\ &\quad -\mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r - \mathbf{g}(\boldsymbol{\eta})]\end{aligned} \tag{23}$$

3.1 Vectorial Backstepping of Marine Vessels

Step 1:

Define the *virtual control*:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \triangleq \mathbf{s} + \boldsymbol{\alpha}_1 \tag{24}$$

where $\mathbf{s}$ is given by Definition 2 and $\boldsymbol{\alpha}_1$ is a *stabilizing function* which can be chosen as:

$$\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\eta}}_r = \dot{\boldsymbol{\eta}}_d - \boldsymbol{\Lambda}\bar{\boldsymbol{\eta}} \tag{25}$$

where $\boldsymbol{\Lambda} > \mathbf{0}$ is a diagonal matrix. Hence (24) can be written:

$$\dot{\bar{\boldsymbol{\eta}}} = -\boldsymbol{\Lambda}\bar{\boldsymbol{\eta}} + \mathbf{s} \tag{26}$$

Consider the Lyapunov function candidate:

$$V_1 = \frac{1}{2}\bar{\boldsymbol{\eta}}^T\mathbf{K}_\eta\bar{\boldsymbol{\eta}} \tag{27}$$

$$\dot{V}_1 = \bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \dot{\bar{\boldsymbol{\eta}}} = -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} + \bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \mathbf{s} \qquad (28)$$

where $\mathbf{K}_\eta = \mathbf{K}_\eta^T > \mathbf{0}$ is a design matrix.

Step 2:

Consider the Lyapunov function candidate:

$$V_2 = V_1 + \frac{1}{2}\mathbf{s}^T \mathbf{M}_\eta(\boldsymbol{\eta})\mathbf{s} \qquad (29)$$

$$\dot{V}_2 = -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} + \mathbf{s}^T[\mathbf{K}_\eta \bar{\boldsymbol{\eta}} + \mathbf{M}_\eta(\boldsymbol{\eta})\dot{\mathbf{s}} + \frac{1}{2}\dot{\mathbf{M}}_\eta(\boldsymbol{\eta})\mathbf{s}] \qquad (30)$$

Substitution of (23) into the expression for $\dot{V}_2$ yields:

$$\begin{aligned} \dot{V}_2 \;=\; & -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} + \mathbf{s}^T[\mathbf{K}_\eta \bar{\boldsymbol{\eta}} + \frac{1}{2}\dot{\mathbf{M}}_\eta(\boldsymbol{\eta})\mathbf{s} \\ & -\mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} - \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})]\mathbf{s} + \mathbf{s}^T \mathbf{J}^{-T}(\boldsymbol{\eta})[\mathbf{Bu} \\ & -\mathbf{M}\dot{\boldsymbol{\nu}}_r - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r - \mathbf{g}(\boldsymbol{\eta})] \qquad (31) \end{aligned}$$

Using the fact that, Fossen [1]:

$$\mathbf{s}^T \left(\frac{1}{2}\dot{\mathbf{M}}_\eta(\boldsymbol{\eta}) - \mathbf{C}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\right)\mathbf{s} = 0, \quad \forall \mathbf{s} \qquad (32)$$

implies that (31) reduces to:

$$\begin{aligned} \dot{V}_2 \;=\; & -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} - \mathbf{s}^T \mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta})\mathbf{s} \\ & +\mathbf{s}^T \mathbf{J}^{-T}(\boldsymbol{\eta})[\mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_\eta \bar{\boldsymbol{\eta}} + \mathbf{Bu} \\ & -\mathbf{M}\dot{\boldsymbol{\nu}}_r - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r - \mathbf{g}(\boldsymbol{\eta})] \qquad (33) \end{aligned}$$

This suggests that the virtual control is chosen according to:

$$\mathbf{Bu} \triangleq \mathbf{z} + \boldsymbol{\alpha}_2 \qquad (34)$$

with stabilizing function:

$$\begin{aligned} \boldsymbol{\alpha}_2 \;=\; & \mathbf{M}\dot{\boldsymbol{\nu}}_r + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{g}(\boldsymbol{\eta}) \\ & -\mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_d \mathbf{s} - \mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_\eta \bar{\boldsymbol{\eta}} \qquad (35) \end{aligned}$$

where $\mathbf{z}$ is a new state variable to be interpreted in Step 3. Hence, the resulting expression for $\dot{V}_2$ takes the form:

$$\begin{aligned} \dot{V}_2 \;=\; & -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} - \mathbf{s}^T(\mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) + \mathbf{K}_d)\mathbf{s} \\ & +\mathbf{s}^T \mathbf{J}^{-T}(\boldsymbol{\eta})\mathbf{z} \qquad (36) \end{aligned}$$

Step 3:

Consider the Lyapunov function candidate:

$$V_3 = V_2 + \frac{1}{2}\mathbf{z}^T \mathbf{z} \qquad (37)$$

$$\begin{aligned} \dot{V}_3 \;=\; & -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} - \mathbf{s}^T(\mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) + \mathbf{K}_d)\mathbf{s} \\ & +\mathbf{z}^T(\mathbf{J}^{-1}(\boldsymbol{\eta})\mathbf{s} + \dot{\mathbf{z}}) \qquad (38) \end{aligned}$$

From (34) we have:

$$\dot{\mathbf{u}} = \mathbf{B}^\dagger(\dot{\mathbf{z}} + \dot{\boldsymbol{\alpha}}_2) \qquad (39)$$

where $\mathbf{B}^\dagger$is the pseudo-inverse:

$$\mathbf{B}^\dagger = \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1} \qquad (40)$$

Hence:

$$\mathbf{T}\dot{\mathbf{u}} = \mathbf{T}\mathbf{B}^\dagger(\dot{\mathbf{z}} + \dot{\boldsymbol{\alpha}}_2) \qquad (41)$$

Substituting the actuator dynamics (17) into this expression yields:

$$\mathbf{u}_c - \mathbf{u} = \mathbf{T}\mathbf{B}^\dagger(\dot{\mathbf{z}} + \dot{\boldsymbol{\alpha}}_2) \qquad (42)$$

which can be solved for $\dot{\mathbf{z}}$ to yield:

$$\dot{\mathbf{z}} = \mathbf{B}\mathbf{T}^{-1}(\mathbf{u}_c - \mathbf{u}) - \dot{\boldsymbol{\alpha}}_2 \qquad (43)$$

Substituting $\dot{\mathbf{z}}$ into (38) yields:

$$\begin{aligned} \dot{V}_3 \;=\; & -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} - \mathbf{s}^T(\mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) + \mathbf{K}_d)\mathbf{s} \\ & +\mathbf{z}^T(\mathbf{J}^{-1}(\boldsymbol{\eta})\mathbf{s} + \mathbf{B}\mathbf{T}^{-1}(\mathbf{u}_c - \mathbf{u}) - \dot{\boldsymbol{\alpha}}_2) \end{aligned}$$

Finally, choosing the *control law* as:

$$\mathbf{u}_c = \mathbf{u} + \mathbf{T}\mathbf{B}^\dagger(\dot{\boldsymbol{\alpha}}_2 - \mathbf{J}^{-1}(\boldsymbol{\eta})\mathbf{s} - \mathbf{K}_z \mathbf{z}) \qquad (44)$$

yields:

$$\dot{V}_3 = -\bar{\boldsymbol{\eta}}^T \mathbf{K}_\eta \boldsymbol{\Lambda} \bar{\boldsymbol{\eta}} - \mathbf{s}^T(\mathbf{D}_\eta(\boldsymbol{\nu},\boldsymbol{\eta}) + \mathbf{K}_d)\mathbf{s} - \mathbf{z}^T \mathbf{K}_z \mathbf{z} \qquad (45)$$

Hence, $\dot{V}_3 < 0 \quad \forall \bar{\boldsymbol{\eta}} \neq \mathbf{0}, \mathbf{s} \neq \mathbf{0}, \mathbf{z} \neq \mathbf{0}$ which according to Lyapunov stability guarantees GES of $\bar{\boldsymbol{\eta}}, \mathbf{s}$ and $\mathbf{z}$.

3.2 Implementation Aspects

The control law (44) is implemented as:

$$\mathbf{u}_c = \mathbf{u} + \mathbf{T}\mathbf{B}^\dagger[\dot{\boldsymbol{\alpha}}_2 - \mathbf{J}^{-1}(\boldsymbol{\eta})\mathbf{s} - \mathbf{K}_z(\mathbf{Bu} - \boldsymbol{\alpha}_2)] \qquad (46)$$

where

$$\begin{aligned} \boldsymbol{\alpha}_2 \;=\; & \mathbf{M}\dot{\boldsymbol{\nu}}_r + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{g}(\boldsymbol{\eta}) \\ & -\mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_d \mathbf{s} - \mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_\eta \bar{\boldsymbol{\eta}} \qquad (47) \end{aligned}$$

The main implementation problem is that $\dot{\boldsymbol{\alpha}}_2$ must be available without using the time derivatives of the states $\boldsymbol{\nu}, \boldsymbol{\eta}$ and $\mathbf{u}$ corresponding to the model (15), (16) and (17). Moreover, only measurements of the states $\boldsymbol{\nu}, \boldsymbol{\eta}$ and $\mathbf{u}$ are assumed.

Exact computation of $\dot{\boldsymbol{\alpha}}_2$ without using the time derivatives of the states: Let us define a smooth vector function:

$$\boldsymbol{\phi}(\boldsymbol{\nu}_r, \boldsymbol{\nu}) \triangleq \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r \tag{48}$$

Hence $\boldsymbol{\alpha}_2$ can be differentiated along the trajectories of $\boldsymbol{\nu}, \boldsymbol{\nu}_r, \dot{\boldsymbol{\nu}}_r, \boldsymbol{\eta}, \mathbf{s}$ and $\bar{\boldsymbol{\eta}}$ according to:

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_2 &= \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{\nu}}\dot{\boldsymbol{\nu}} + \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{\nu}_r}\dot{\boldsymbol{\nu}}_r + \frac{\partial \boldsymbol{\alpha}_2}{\partial \dot{\boldsymbol{\nu}}_r}\ddot{\boldsymbol{\nu}}_r \\ &\quad + \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{\eta}}\dot{\boldsymbol{\eta}} + \frac{\partial \boldsymbol{\alpha}_2}{\partial \mathbf{s}}\dot{\mathbf{s}} + \frac{\partial \boldsymbol{\alpha}_2}{\partial \bar{\boldsymbol{\eta}}}\dot{\bar{\boldsymbol{\eta}}} \end{aligned} \tag{49}$$

Since $\frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{\nu}} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}}$, we can write:

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_2 &= \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}}\dot{\boldsymbol{\nu}} + (\mathbf{C}(\boldsymbol{\nu}) + \mathbf{D}(\boldsymbol{\nu}))\dot{\boldsymbol{\nu}}_r + \mathbf{M}\ddot{\boldsymbol{\nu}}_r \\ &\quad + \frac{\partial \mathbf{g}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\dot{\boldsymbol{\eta}} - \frac{\partial \mathbf{J}^T(\boldsymbol{\eta})(\mathbf{K}_d\mathbf{s} + \mathbf{K}_\eta\bar{\boldsymbol{\eta}})}{\partial \boldsymbol{\eta}}\dot{\boldsymbol{\eta}} \\ &\quad - \mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_d\dot{\mathbf{s}} - \mathbf{J}^T(\boldsymbol{\eta})\mathbf{K}_\eta\dot{\bar{\boldsymbol{\eta}}} \end{aligned} \tag{50}$$

The expression:

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} = \frac{\partial[\mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu}_r]}{\partial \boldsymbol{\nu}} \tag{51}$$

depends on the structure of the model matrices $\mathbf{C}(\boldsymbol{\nu})$ and $\mathbf{D}(\boldsymbol{\nu})$. If $\mathbf{C}$ and $\mathbf{D}$ are constant (linear theory), we simply get:

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} = \mathbf{0} \tag{52}$$

The time derivatives in (50) are computed according to:

$$\begin{aligned} \dot{\boldsymbol{\nu}} &= \mathbf{M}^{-1}[\mathbf{B}\mathbf{u} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})] && (53) \\ \dot{\boldsymbol{\eta}} &= \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} && (54) \\ \ddot{\boldsymbol{\eta}} &= \dot{\mathbf{J}}(\boldsymbol{\eta})\boldsymbol{\nu} + \mathbf{J}(\boldsymbol{\eta})\dot{\boldsymbol{\nu}} && (55) \end{aligned}$$

which are substituted into:

$$\begin{aligned} \dot{\bar{\boldsymbol{\eta}}} &= \dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\eta}}_d && (56) \\ \ddot{\bar{\boldsymbol{\eta}}} &= \ddot{\boldsymbol{\eta}} - \ddot{\boldsymbol{\eta}}_d && (57) \\ \dot{\mathbf{s}} &= \ddot{\bar{\boldsymbol{\eta}}} + \boldsymbol{\Lambda}\dot{\bar{\boldsymbol{\eta}}} && (58) \\ \dot{\boldsymbol{\nu}}_r &= \mathbf{J}^{-1}(\boldsymbol{\eta})[\ddot{\boldsymbol{\eta}}_r - \dot{\mathbf{J}}(\boldsymbol{\eta})\boldsymbol{\nu}_r] && (59) \\ \ddot{\boldsymbol{\nu}}_r &= -\mathbf{J}^{-1}\dot{\mathbf{J}}(\boldsymbol{\eta})\mathbf{J}^{-1}(\boldsymbol{\eta})[\ddot{\boldsymbol{\eta}}_r - \dot{\mathbf{J}}(\boldsymbol{\eta})\boldsymbol{\nu}_r] \\ &\quad + \mathbf{J}^{-1}(\boldsymbol{\eta})[\boldsymbol{\eta}_r^{(3)} - \ddot{\mathbf{J}}(\boldsymbol{\eta})\boldsymbol{\nu}_r - \dot{\mathbf{J}}(\boldsymbol{\eta})\dot{\boldsymbol{\nu}}_r] && (60) \end{aligned}$$

where $\boldsymbol{\eta}_r^{(3)} = \boldsymbol{\eta}_d^{(3)} - \boldsymbol{\Lambda}(\ddot{\boldsymbol{\eta}} - \ddot{\boldsymbol{\eta}}_d)$. An attractive simplification of this scheme is given below:

Approximative solution of $\dot{\boldsymbol{\alpha}}_2$ without using the time derivatives of the states: For marine vessels a good approximation is:

$$\dot{\mathbf{J}}(\boldsymbol{\eta}) = \ddot{\mathbf{J}}(\boldsymbol{\eta}) = \mathbf{0} \tag{61}$$

In fact this is a good approximation since the dynamics of a marine vessel is quite slow due to the dissipative nature of hydrodynamic damping and the relatively large inertia of the vessel. Typical values for the bandwidth of a ship is 0.01–0.1 (rad/s). Hence, the assumption (61) is common in most industrial ship control systems. This implies that Eqns. (53)–(60) reduces to:

$$\begin{aligned} \dot{\boldsymbol{\nu}} &= \mathbf{M}^{-1}[\mathbf{B}\mathbf{u} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})] && (62) \\ \dot{\mathbf{s}} &= \mathbf{J}(\boldsymbol{\eta})\mathbf{M}^{-1}[\mathbf{B}\mathbf{u} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})] \\ &\quad - \ddot{\boldsymbol{\eta}}_d + \boldsymbol{\Lambda}(\mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} - \dot{\boldsymbol{\eta}}_d) && (63) \\ \dot{\boldsymbol{\nu}}_r &= \mathbf{J}^{-1}(\boldsymbol{\eta})[\ddot{\boldsymbol{\eta}}_d - \boldsymbol{\Lambda}(\mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} - \dot{\boldsymbol{\eta}}_d)] && (64) \\ \ddot{\boldsymbol{\nu}}_r &= \mathbf{J}^{-1}(\boldsymbol{\eta})[\boldsymbol{\eta}_d^{(3)} - \boldsymbol{\Lambda}\mathbf{J}(\boldsymbol{\eta})\mathbf{M}^{-1}[\mathbf{B}\mathbf{u} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} \\ &\quad - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})] + \boldsymbol{\Lambda}\ddot{\boldsymbol{\eta}}_d] && (65) \end{aligned}$$

4 Simulation of a Nonlinear Supply Vessel

Consider a supply vessel with mass $m = 6.4 \cdot 10^6$ (kg) and length $L = 76.2$ (m) given by the following non-dimensional matrices (Bis-scaling) [1]:

Figure 1: Supply vessel used in the simulation study.

$$\mathbf{M}'' = \begin{bmatrix} 1.1274 & 0 & 0 \\ 0 & 1.8902 & -0.0744 \\ 0 & -0.0744 & 0.1278 \end{bmatrix}$$

$$\mathbf{C}''(\boldsymbol{\nu}'') = \begin{bmatrix} 0 & 0 & -1.8902v''+0.0744r'' \\ 0 & 0 & 1.1274u'' \\ 1.8902v''-0.0744r'' & -1.1274u'' & 0 \end{bmatrix}$$

$$\mathbf{D}'' = \begin{bmatrix} 0.0414 & 0 & 0 \\ 0 & 0.1775 & -0.0141 \\ 0 & -0.1073 & 0.0568 \end{bmatrix}$$

$$\mathbf{J}''(\boldsymbol{\eta}'') = \begin{bmatrix} \cos\psi'' & -\sin\psi'' & 0 \\ \sin\psi'' & \cos\psi'' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}'' = 5.0\sqrt{g/L}\cdot\mathbf{I}_{6\times6} = 1.7964\cdot\mathbf{I}_{6\times6}$$

$$\mathbf{B}'' = 10^{-3}\begin{bmatrix} 13.0 & 13.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11.6 & 11.6 & 6.0 & 6.7 \\ 0 & 0 & -4.6 & -4.6 & 2.7 & 2.2 \end{bmatrix}$$

where $\boldsymbol{\nu}'' = [u'', v'', r'']^T$ and $\boldsymbol{\eta}'' = [x'', y'', \psi'']^T$. The thruster inputs $\mathbf{u}'' = [u''_1, ..., u''_6]^T$ are scaled such that $|u''_i| \leq 1.0$ (i=1...6). The actuator time constants are chosen as 5.0 (s) for all the thrusters and propellers. The non-dimensional state vectors are transformed to physical quantities by:

$$\begin{aligned} \boldsymbol{\eta} &= \mathrm{diag}\{L, L, 1\}\boldsymbol{\eta}'' \\ \boldsymbol{\nu} &= \mathrm{diag}\{\sqrt{gL}, \sqrt{gL}, \sqrt{g/L}\}\boldsymbol{\nu}'' \end{aligned}$$

The backstepping control law (46)–(47) and (62)–(65) was simulated on a computer with $\mathbf{K}_d = \mathbf{K}_\eta = \mathbf{I}_{3\times3}, \boldsymbol{\Lambda} = 0.1\cdot\mathbf{I}_{3\times3}$ and non-dimensional sampling time $h'' = 0.02$ corresponding to $h = 0.0557$ (s). The simulation results are shown in Figure 1 where the control law and the reference model is started at $t = 15$ (s). Even though, $\dot{\mathbf{J}}(\boldsymbol{\eta})$ and $\ddot{\mathbf{J}}(\boldsymbol{\eta})$ are assumed to be zero in the computation of $\dot{\boldsymbol{\alpha}}_2$, excellent tracking results are obtained when using a 3rd-order filtered reference trajectory. The steady-state values of the reference model were chosen as:

$$\lim_{t\to\infty} \boldsymbol{\eta}_d(t) = [10\ (\mathrm{m}), -5\ (\mathrm{m}), 10\ (\mathrm{deg})]^T. \quad (66)$$

Also notice that all non-dimensional control inputs u''_i (i=1...6) are inside their maximum values of ±1.0.

5 Conclusions

In this paper a *vectorial backstepping* control law for conventional ships is derived with emphasizes placed on the actuator dynamics. The proposed control law is simulated on a supply vessel with excellent results.

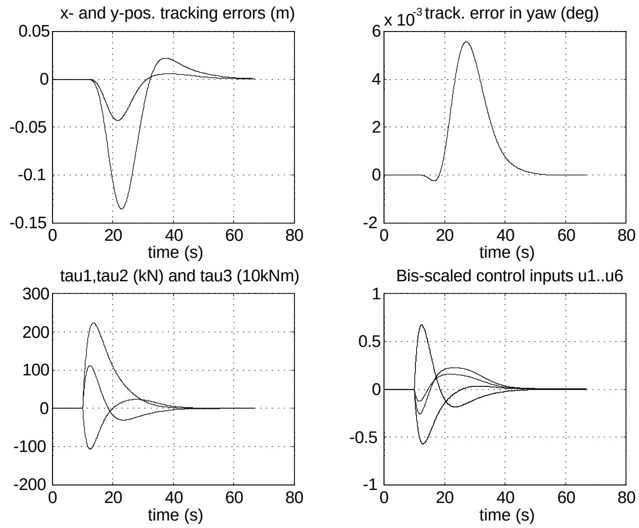

Figure 2: *Upper left:* x- and y-position tracking errors. *Upper right:* yaw angle tracking error. *Lower left:* $\boldsymbol{\tau}_c = \mathbf{B}\mathbf{u}_c$. *Lower right:* non-dimensional controls u''_i (i=1...6). Notice that $u_1 = u_2$ and $u_3 = u_4$.

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