Stabilization of Integrator Chains in the Presence of Magnitude and Rate Saturations; a Gain Scheduling Approach*

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Abstract

In this paper we present a gain scheduling technique for stabilizing integrator chains in the presence of magnitude and rate limitations on the input. The controller was first presented in [4] where it showed excellent experimental results on a pitch axis flight control experiment at Caltech in the case of input rate limits only. The controller has several interesting features, e.g. it is easy to tune and it gives a minimal loss in performance. These properties are illustrated in a simulation study.

1 Introduction

Actuator magnitude and rate saturations are some of the most common and significant nonlinearities in control systems. These nonlinearities are present in virtually all physical systems, even though the degree of importance may vary. In some control systems the input nonlinearities may be neglected, while in other their effects may considerable, leading to stability problems or aggravation of performance if ignored in control design. At present, there are few systematic means of analyzing and designing nonlinear control systems in the presence of magnitude and rate saturations.

One example of application where input magnitude and, in particular, rate saturations have significant effect are flight control of high agility aircrafts and rudder roll stabilization of ships. In flight control saturating actuators influence the stability of the aircraft. In fact, the YF-22 crash of April, 1992, has been blamed on a pilot-induced oscillation (PIO) caused in part by rate saturated control surfaces, see [1] and [3] for a detailed description. This example illustrates the need for new tools dealing with input nonlinearities.

On the practical side, input magnitude and rate saturations are often handled in an *ad hoc* fashion and the stability properties of a control system is investigated through extensive simulations before implementation. Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22), stronger theoretical understanding is required as the demands for performance and agility are increasing.

In the linear literature there exists results allowing actuator saturations to be incorporated in the design process. An example is the use of l_1 analysis and synthesis techniques, see e.g.[2], The drawback with these approaches is that linear controllers will be generated for linear systems. Hence, like any other method for linear control design, the control law will in general be very conservative since the control gains must be small enough to tolerate the worst case scenario.

There have been several recent results in the nonlinear control literature dealing with saturations that shed some new light to the problem. Teel introduced the use of nested saturations for chains of integrators [8] and this has been generalized to include much more general systems, see e.g. [7]. More closely related to the work presented here are the results of Megretski [6] and Teel [9].

In this paper we take a gain scheduling approach to the problem of stabilizing integrator chains in the presence of magnitude and rate saturations. The algorithm was first presented in [4] for input rate saturations only, and the gain scheduled control law showed excellent experimental results when implemented on a pitch axis flight control experiment at Caltech. In this paper we generalize the results in [4] and show that the nonlinear and time-varying controller give a closed-loop system which is provably convergent in the presence of simultaneous magnitude and rate saturations.

2 Problem Statement

The problem we consider in this paper is to stabilize an integrator chain of length n. The chain is written in the compact form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector and $u \in \mathcal{U}_1 \subset \mathbb{R}$ is the input to the system. The input is subject to magnitude and rate saturations, and thus the input dynamics can be written in the form

$$u = \sigma_1(v) \tag{2}$$

$$\dot{v} = \sigma_2\left(\frac{1}{\tau}(u_c - v)\right), \tag{3}$$

where $v \in \mathbb{R}$ is an internal state of the input dynamics and $u_c \in \mathbb{R}$ is the commanded input. The functions $\sigma_i : \mathbb{R} \to \mathcal{U}_i$, $i = 1, 2$, where $\mathcal{U}_i = [-\beta_i, \beta_i]$, are saturation functions defined as

$$\sigma_i(\cdot) = \mathrm{sign}(\cdot)\min(|\cdot|, \beta_i), \quad \beta_i > 0, \quad i = 1, 2. \tag{4}$$

The goal of the control design in this paper is not only to make the closed-loop system convergent, but also to obtain a control system with good performance. Hence, the closed-loop system should have

*This research was funded in part by AFOSR grant F49620-95-1-0419 and ONR grant N00014-96-1-0804

- A rise time as close as possible to the rise time of the linear system with linear controller.
- No large overshoot compared to the response of the linear system with linear controller (also called the "anti-windup" problem).

3 Control Law

The approach we take in this paper is gain scheduling of a linear control law. The method used is given in the following definition.

Definition 1 Let $u_c = -\boldsymbol{g}^T\boldsymbol{x}$ be a linear control law stabilizing the pair $(\boldsymbol{A}, \boldsymbol{b})$ when no input dynamics are present. Then the *gain scheduled control law* is defined to be

$$u_c(t, \boldsymbol{x}) \triangleq -\boldsymbol{g}^T\boldsymbol{K}(\gamma(t))\boldsymbol{x}, \qquad (5)$$

where

$$\boldsymbol{K}(\gamma(t)) = \mathrm{diag}(\gamma^n(t), \cdots, \gamma(t)), \qquad (6)$$

and $\gamma(t) \in (0, 1]$ is the *scaling factor.* $\triangle$

The scaling factor is given by the following definition.

Definition 2 Let $h(t)$ be a piecewise continuous function of time given by

$$h(t) = 0 \text{ if } u = u_c, \quad \dot{h}(t) = 1 \text{ otherwise.} \qquad (7)$$

where u and u_c are the actual and commanded input, respectively. Let T_k denote the time instance when $u = u_c$ for the kth time, $\gamma_k = \gamma(T_k)$, $\gamma_0 = 1$, N a positive number and $g(t) = (t - T_N)$. Define $\gamma(t)$ as

$$\gamma(t) \triangleq \begin{cases} \gamma_k & \text{if } 0 < h(t) \leq \dfrac{\Delta T}{\gamma_k^{1/p}}, \\ \left(\dfrac{\Delta T}{h(t)}\right)^p & \text{if } k \leq N \text{ and } h(t) > \dfrac{\Delta T}{\gamma_k^{1/p}}, \\ \left(\dfrac{\Delta T}{g(t)}\right)^q & \text{if } k > N \text{ and } g(t) > \dfrac{\Delta T}{\gamma_N^{1/q}}, \end{cases} \qquad (8)$$

where $p \geq 0$, $q > 1$ and $\Delta T > 0$. $\triangle$

The control law given by Definitions 1 and 2 can be shown to give a convergent closed-loop system. For proof of stability, see [5].

4 Illustrative Example

Consider a chain of three integrators

$$\dot{x}_1 = x_2, \ \ \dot{x}_2 = x_3, \ \ \dot{x}_3 = u \qquad (9)$$

where the input is subject to magnitude and rate constraints of unity value ($\beta_1 = \beta_2 = 1$). The nonlinear scheduled control law is then given by

$$u_c(t, \boldsymbol{x}) = -\gamma^3(t)x_1 - 3\gamma^2(t)x_2 - 3\gamma(t)x_3, \qquad (10)$$

where $\gamma(t)$ is given by Definition 2 with $\Delta T = 0.5$ (s) and $p = 1.01$. The simulation results are shown in Figure 1, where the dashed line show the response of the linear system with the linear controller ($\gamma(t) \equiv 1$) while the solid line show the response of the nonlinear system with the nonlinear controller. Note that the nonlinear system with the linear controller is unstable for this step input.

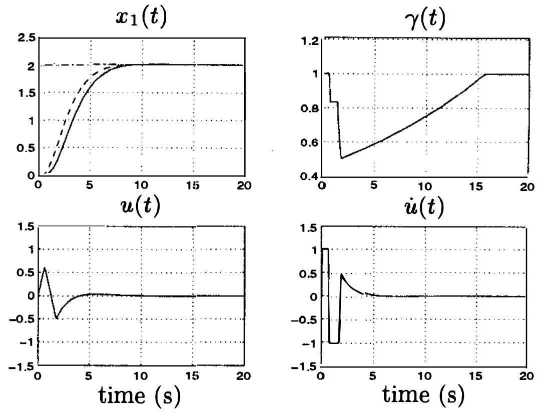

Figure 1: The time response for $\boldsymbol{x}(0) = [0, 0, 0]^T$ and $\boldsymbol{x}_d = [2, 0, 0]^T$.

5 Conclusions

In this paper we have given a time-varying controller for stabilizing a chain of integrators in the presence of magnitude and rate saturations. It is proved that the controller gives a convergent closed-loop system, i.e. for any bounded state initial condition the state will converge to the origin. The main strength of the controller is that it gives provable convergence without being overly conservative. This is verified in a simulation study.

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