Stabilization of Linear Unstable Systems with Control Constraints

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Abstract

In this paper we consider linear systems with possibly exponentially unstable eigenvalues and with saturating input. It is shown that for a class of unstable systems there exists a bounded linear stabilizing controller for sets of initial conditions which may be arbitrary large and bounded in some directions of the state space while other directions must be bounded. Hence the results are stronger than the existing local stability results and weaker than semi-global stability (impossible to obtain for unstable systems). Moreover, sufficient conditions for the existence of stabilizing controllers when the system is subject to plant disturbances and measurement noise are also given.

1 Introduction

The last few years there have been an extensive interest in stabilization of linear systems subject to actuator saturation, and several significant results have emerged. The main focus have been on global and semi-global stabilization of linear systems, and in particular null-controllable systems. Null-controllable systems have, in addition to the usual stabilizability property, eigenvalues with non-positive real part, see e.g. [7].

In general, null-controllable systems are not globally stabilizable by bounded linear control laws. This fact was pointed out in [2]. However, in [10] it was shown that an integrator chain of length n can be globally stabilized by bounded input using nonlinear controllers. This result was later generalized for all null-controllable systems in [9].

The semi-global framework is useful for systems with saturating actuators, since many control problems are concerned with a bounded region of attraction rather than the full state space. It was shown in [5] that continuous time linear null-controllable systems can be semi-globally stabilized by linear control laws, i.e. there exists a bounded control law stabilizing the system for any arbitrary large and bounded set of initial conditions. This result was later elaborated on in e.g. [6] and [11].

The above mentioned results are all based on the assumption that the open-loop system is null-controllable, and gives only local results for exponentially unstable systems. Hence, a set of initial conditions is valid for stabilization with bounded control only if it is contained in a bounded ball around the origin with radius a function of the input limit. The reason for the lack of other than local results is that global and semi-global stability is not possible when the system has poles with positive real part. Alternative approaches discussing stabilization of unstable systems with bounded control, see e.g. [14], [8], [3],[1].

In this paper we show that for a class of unstable systems it is possible to obtain an arbitrary large domain of attraction in certain directions of the state space while other directions are upper bounded with bounds depending on the input constraints. This approach is appealing, since e.g. mechanical systems have natural "bounds" on the velocities while the position may be practically unbounded. Moreover, the initial condition of the state is not required to be upper bounded, but the directions where it may be increased arbitrarily are restricted. The proposed stability results in this paper can thus be viewed as something between local and semi-global stability.

Throughout this paper, $\mathcal{X}$ will denote a set of initial conditions for a vector $\boldsymbol{x}$. Writing $\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T$, $\mathcal{X}_1$ and $\mathcal{X}_2$ denotes sets of initial conditions for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. The notation $\|\cdot\|$ will be used both for the Euclidean vector norm and the induced matrix norm. $\mathbb{C}$ and $\mathbb{C}^+$ will denote the complex plane and the open right half of the complex plane, respectively. For a positive definite matrices, $\boldsymbol{P}$, the matrix square root exists and will be denoted $\boldsymbol{P}^{1/2}$, that is $\boldsymbol{P} = \boldsymbol{P}^{1/2}\boldsymbol{P}^{1/2}$. The eigenvalues of a matrix $\boldsymbol{A}$ is denoted $\lambda(\boldsymbol{A})$ and the boundary of a set $\mathcal{B}$ is denoted $\partial\mathcal{B}$.

2 Problem Statement

The problem we consider is stabilization of multi-input multi-output (MIMO) systems with input magnitude saturation. It is assumed full state feedback, with process noise and disturbances on the state measurements. Hence, the systems can be written in the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\sigma(\boldsymbol{u}(t)) + \boldsymbol{E}\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{x}(t) + \boldsymbol{v}(t), \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^m$ is the commanded input, $\boldsymbol{w} \in \mathbb{R}^n$ is the plant disturbance vector, $\boldsymbol{y} \in \mathbb{R}^n$ is the state measurement vector, $\boldsymbol{v} \in \mathbb{R}^n$ is the measurement noise vector, and the map $\sigma : \mathbb{R}^m \to \mathcal{U} \subset \mathbb{R}^m$ is a saturation function given by the following definition.

Definition 1 (Saturation function) A function $\boldsymbol{\sigma}$: $\mathbb{R}^m \to \mathcal{U} \subset \mathbb{R}^m$ is called a saturation function if

1. $\boldsymbol{\sigma}(\boldsymbol{u})$ is decentralized, i.e.

$$\boldsymbol{\sigma}(\boldsymbol{u}) = [\ \sigma_1(u_1)\ \ \sigma_2(u_2)\ \ \cdots\ \ \sigma_m(u_m)\ ]^T. \tag{2}$$

2. There exists constants $\Delta_i > 0$, $i = 1, \cdots, n$, such that

$$\begin{array}{ll} u_i\sigma_i(u_i) \geq u_i^2, & \text{for } |u_i| \leq \Delta_i, \\ |\sigma_i(u_i)| \geq \Delta_i, & \text{for } |u_i| > \Delta_i. \end{array} \tag{3}$$

3. σ_i is locally Lipschitz. $\triangle$

Throughout the paper we will use Δ defined by

$$\Delta \triangleq \min_i \Delta_i \tag{4}$$

It is well-known that, in general, all linear systems with input saturation are not globally or semi-globally stabilizable by linear feedback. Most of the existing stability results assume that the open-loop system is null-controllable, i.e. that the systems are stabilizable and have eigenvalues with non-positive real part. Clearly, systems having one or more eigenvalues with positive real part are not null-controllable, and it is impossible to semi-globally stabilize such systems.

In this paper systems having eigenvalues with non-positive real part are also considered. Thus, we do not assume null-controllability, but make the following assumption throughout the paper.

Assumption 1 The pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable. $\triangle$

The results we will derive for unstable systems refers to a specific class of systems. Consider rewriting the state vector in (1) such that

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1^T, & \boldsymbol{x}_2^T \end{bmatrix}^T, \tag{5}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^q$ and $\boldsymbol{x}_2 \in \mathbb{R}^{n-q}$. Then, the system dynamics (1) can be rewritten

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix} \sigma(\boldsymbol{u}). \tag{6}$$

with obvious dimensions of the matrices $\boldsymbol{A}_{ij}$, $i, j = 1, 2$, and $\boldsymbol{B}_i$, $i = 1, 2$. Using (6) we consider a class of systems given by the following definition.

Definition 2 Let $\mathcal{M}_q^{n\times n} \subset \mathbb{R}^{n\times n}$ be such that any matrix $\boldsymbol{A} \in \mathcal{M}_q^{n\times n}$ satisfy the following properties:

1. $\lambda(\boldsymbol{A}_{11}) \in \mathbb{C} \setminus \mathbb{C}^+$,
2. $\lambda(\boldsymbol{A}_{22}) \in \mathbb{C}^+$,
3. $\boldsymbol{A}_{21} = \boldsymbol{0}$. $\triangle$

Remark 1 Notice that if $\boldsymbol{A} \in \mathcal{M}_n^{n\times n}$, then $\boldsymbol{A} = \boldsymbol{A}_{11}$. Hence, all eigenvalues of $\boldsymbol{A}$ have non-positive real parts and, by Assumption 1 the system is null-controllable. If $\boldsymbol{A} \in \mathcal{M}_0^{n\times n}$ then $\boldsymbol{A} = \boldsymbol{A}_{22}$ and all eigenvalues of $\boldsymbol{A}$ have positive real parts. $\triangle$

Control systems with a system matrix $\boldsymbol{A} \in \mathcal{M}_q^{n\times n}$ have some interesting properties given in Section 4. Before we state and prove these properties, some preliminary results, given in the preceding section, are needed.

3 Preliminaries

In this section we consider the system (1) without disturbances, i.e. it is assumed that $\boldsymbol{w}(t) = \boldsymbol{v}(t) = 0$.

Lemma 1 Let $\sigma : \mathbb{R}^m \to U$ be a saturation function and let $\mathcal{Y}_\Delta^- \subseteq \mathcal{Y}_\Delta^+ \subset \mathbb{R}^m$ be given by

$$\mathcal{Y}_\Delta^+ = \{\boldsymbol{y} \in \mathbb{R}^m \ : \ \|\boldsymbol{y}\| \leq \Delta + s\}. \tag{7}$$

$$\mathcal{Y}_\Delta^- = \{\boldsymbol{y} \in \mathbb{R}^m \ : \ \|\boldsymbol{y}\| \leq \Delta - s\}. \tag{8}$$

where $s \in [0, \Delta]$. Then, the function

$$f(\boldsymbol{a}, \boldsymbol{b}) \triangleq \begin{cases} \kappa, & \text{if } \sigma(\kappa\boldsymbol{a}+\boldsymbol{b}) = \kappa(\boldsymbol{a}+\boldsymbol{b}) \\ \dfrac{\boldsymbol{a}^T\sigma(\kappa(\boldsymbol{a}+\boldsymbol{b}))}{\boldsymbol{a}^T(\boldsymbol{a}+\boldsymbol{b})}, & \text{otherwise} \end{cases} \tag{9}$$

is continuous and satisfy

$s = \Delta$: $f(\boldsymbol{a}, \boldsymbol{0}) \geq 1/2$, $\forall\, \boldsymbol{a} \in \mathcal{Y}_\Delta^+$, $\forall\, \kappa \in [1, \infty)$.

$s < \Delta$: $f(\boldsymbol{a}, \boldsymbol{b}) \geq 1/2$, $\forall\, \boldsymbol{a} \in \mathcal{Y}_\Delta^+$, $\forall\, \boldsymbol{b} \in \mathcal{Y}_\Delta^-$, $\kappa = 1$.

Proof: First, continuity of $f(\boldsymbol{a}, \boldsymbol{b})$ with respect to $\boldsymbol{a}$ and $\boldsymbol{b}$ follows trivially by its definition.

Let $\boldsymbol{b} \equiv \boldsymbol{0}$ and using the fact that the saturation function is decentralized, we have

$$\boldsymbol{a}^T\sigma(\kappa\boldsymbol{a}) = \sum_{i=1}^n a_i\sigma_i(\kappa a_i). \tag{10}$$

Using the properties of σ_i and taking $s = \Delta$ in (7) we get

$$a_i\sigma_i(\kappa a_i) \geq \frac{1}{2}a_i^2, \ \ \forall\, \kappa \in [1, \infty) \tag{11}$$

and if follows that

$$\boldsymbol{a}^T\sigma(\kappa\boldsymbol{a}) \geq \frac{1}{2}\boldsymbol{a}^T\boldsymbol{a}, \ \ \forall \boldsymbol{a} \in \mathcal{Y}_\Delta^+. \tag{12}$$

Hence, $f(\boldsymbol{a}, \boldsymbol{0}) \geq 1/2$, $\forall\, \boldsymbol{a} \in \mathcal{Y}_\Delta^+$, $\forall\, \kappa \in [1, \infty)$.

Next, for $\boldsymbol{b} \in \mathcal{Y}_\Delta^-$ and $\kappa = 1$, we have

$$\boldsymbol{a}^T\sigma(\boldsymbol{a}+\boldsymbol{b}) = \sum_{i=1}^n a_i\sigma_i(a_i + b_i). \tag{13}$$

The definition of σ_i and (7) implies

$$a_i\sigma_i(a_i + b_i) \geq \frac{1}{2}(a_i^2 + a_ib_i), \tag{14}$$

whenever $a_i\sigma_i(a_i + b_i) \geq 0$. But if $a_i\sigma_i(a_i + b_i) < 0$ then $|a_i + b_i| \leq \Delta$ and it follows that

$$\frac{\boldsymbol{a}^T\sigma(\boldsymbol{a}+\boldsymbol{b})}{\boldsymbol{a}^T(\boldsymbol{a}+\boldsymbol{b})} \geq \frac{1}{2}, \ \ \forall \boldsymbol{a} \in \mathcal{Y}_\Delta^+. \tag{15}$$

Hence, $f(\boldsymbol{a}, \boldsymbol{b}) \geq 1/2$ and this completes the proof. $\square$

Lemma 1 states that the function $f(\boldsymbol{a}, \boldsymbol{b})$ is bounded from below by $1/2$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy some conditions. Moreover, it is shown that this bound is independent of κ. This result will show to be very useful when dealing with bounded input systems controlled by optimal control laws.

The control laws we consider in this paper is linear and in the form

$$u = -\kappa\boldsymbol{B}^T\boldsymbol{P}(\gamma)\boldsymbol{x}, \tag{16}$$

where $\kappa \geq 1$ is a constant design parameter. The matrix $\boldsymbol{P}(\gamma)$ is the solution of the Algebraic Riccati Equation (ARE):

$$\boldsymbol{A}^T\boldsymbol{P}(\gamma) + \boldsymbol{P}(\gamma)\boldsymbol{A} - \boldsymbol{P}(\gamma)\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P}(\gamma) = -\boldsymbol{Q}(\gamma), \tag{17}$$

where $Q(\gamma)$ is positive definite for all $\gamma \in (0,1]$ and satisfy

$$\lim_{\gamma \to 0} Q(\gamma) = \mathbf{0}. \tag{18}$$

The solution of (17) has the important property that it approaches the zero matrix as $\gamma \to 0$ if the system is null-controllable. This was first pointed out in [6] and is, for completeness, given in the following lemma.

Lemma 2 Consider the system (1) with $A \in \mathcal{M}_n^{n\times n}$, and let $P(\gamma)$ be the solution of (17). Then,

$$\lim_{\gamma \to 0} P(\gamma) = \mathbf{0}, \tag{19}$$

and $P(\gamma)$ is positive definite $\forall \gamma \in (0,1]$.

Proof: By Assumption 1, the system is null-controllable and a proof can be found in [6] and the references therein. □

Lemma 2 guarantees that

$$u = -B^T P(\gamma) x \tag{20}$$

is a stabilizing control law for (1) for all $\gamma \in (0,1]$. This result is used in [6] to semi-globally stabilize any null-controllable systems.

Since this paper addresses unstable systems, the question is what happens with the limit of $P(\gamma)$, denoted $\overline{P}$ when $q < n$. Clearly, in this case it is not possible that $P(\gamma)$ approach the zero matrix. In order to find the limit, let $\overline{P}_{22} \in \mathbb{R}^{(n-q)\times(n-q)}$ be the solution of the ARE

$$A_{22}^T \overline{P}_{22} + \overline{P}_{22} A_{22} - \overline{P}_{22} B_2 B_2^T \overline{P}_{22} = \mathbf{0}. \tag{21}$$

Remark 2 Since all eigenvalues of A_{22} have positive real parts there is no unique solution to this equation. This follows easily from the fact that the zero matrix is one solution to (21). However, there is exactly one solution (see [13]) with the property that

$$\mathrm{Re}\left\{\lambda\left(A_{22} - B_2 B_2^T \overline{P}_{22}\right)\right\} < 0. \tag{22}$$

Thus, when referring to $\overline{P}_{22}$, we refer to the matrix satisfying (22). △

Then, in analogy to Lemma 2, we state the following lemma:

Lemma 3 Consider the system (1) with $A \in \mathcal{M}_q^{n\times n}$, and let $P(\gamma)$ be the solution of (17). Then,

$$\lim_{\gamma \to 0} P(\gamma) = \overline{P} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{P}_{22} \end{bmatrix} \tag{23}$$

where $P(\gamma)$ is positive definite $\forall \gamma \in (0,1]$ and $\overline{P}_{22}$ is the solution of (21).

Proof: Since the pair (A, B) is stabilizable, it is well-known that (17) has at least one solution for any $\gamma \in (0,1]$. From [13] we know that if (17) has solutions, then there is exactly one solution, denoted $P^+(\gamma)$, with the property

$$\mathrm{Re}\left\{\lambda\left(A - BB^T P^+(\gamma)\right)\right\} \leq 0. \tag{24}$$

Any other solution, $P(\gamma)$, is bounded from above by $P^+(\gamma)$, i.e.

$$P(\gamma) \leq P^+(\gamma). \tag{25}$$

Continuity of $P^+(\gamma)$ at $\gamma = 0$ has been shown in [12], and implies

$$\lim_{\gamma \to 0} P^+(\gamma) = P^+(0). \tag{26}$$

Hence, to prove the lemma, it is sufficient to show $\overline{P} = P^+(0)$ or, equivalently, that (i) $\overline{P}$ is a solution of (17) and that (ii) (24) is satisfied when $P^+(\gamma)$ is replaced by $\overline{P}$.

(i) Inserting $\overline{P}$ given by (23) into (17) and using (21) we obtain

$$A^T \overline{P} + \overline{P} A - \overline{P} B B^T \overline{P} = \mathbf{0}, \tag{27}$$

which shows that $\overline{P}$ is indeed a solution.

(ii) Using the identity

$$A - BB^T\overline{P} = \begin{bmatrix} A_{11} & A_{12} - B_1 B_2{}^T \overline{P}_{22} \\ \mathbf{0} & A_{22} - B_2 B_2{}^T \overline{P}_{22} \end{bmatrix}, \tag{28}$$

(22) implies

$$\mathrm{Re}\left\{\lambda\left(A - BB^T\overline{P}\right)\right\} \leq 0. \tag{29}$$

Hence, $\overline{P} = P^+(0)$ and this concludes the proof. □

The controller derived by decreasing γ sufficiently is low-gain controllers. In some control systems it might be desirable to increase the control gains in order to obtain e.g. disturbance attenuation, and the following lemma show that if a low-gain controller can be found, a high-gain controller will also give a stable closed-loop system.

Lemma 4 Consider the system (1) and assume that (17) has a solution, $P(\gamma)$, such that

$$\|B^T P^{\frac{1}{2}}(\gamma)\| \, \|P^{\frac{1}{2}}(\gamma) x\| \leq 2\Delta, \;\; \forall x(t_0) \in \mathcal{X}, \tag{30}$$

Then the closed-loop system

$$\dot{x} = Ax - B\sigma\left(\kappa B^T P(\gamma) x\right), \;\; x(t_0) \in \mathcal{X} \tag{31}$$

is asymptotically stable for any $\kappa \geq 1$.

Proof: Let $P = P(\gamma)$ be the positive definite solution of (17) satisfying (30). Consider the change of coordinates

$$z = P^{\frac{1}{2}} x, \tag{32}$$

and define

$$a \triangleq B^T P^{\frac{1}{2}} z. \tag{33}$$

Then $z \in \mathcal{B}_\Delta$ and $a \in \mathcal{Y}_\Delta^+$ where

$$\mathcal{B}_\Delta = \left\{ z \in \mathbb{R}^n \;:\; \|B^T P^{\frac{1}{2}}(\gamma)\| \, \|z\| \leq 2\Delta \right\}, \tag{34}$$

and $\mathcal{Y}_\Delta^+$ is given by (7) with $s = \Delta$. Consider the following Lyapunov function candidate

$$V = x^T P x = z^T z, \tag{35}$$

with time derivative given by

$$\dot{V} = z^T P^{-\frac{1}{2}} \left(A^T P + PA\right) P^{-\frac{1}{2}} z$$
$$-2z^T P^{\frac{1}{2}} B\sigma\left(\kappa B^T P^{\frac{1}{2}} z\right), \quad z(t) \in \mathcal{B}_\Delta, \quad (36)$$

Since $a = B^T P^{\frac{1}{2}} z \in \mathcal{Y}_\Delta^+$ for all $z \in \mathcal{B}_\Delta$, we can rewrite (36), yielding

$$\dot{V} = z^T P^{-\frac{1}{2}} \left(A^T P + PA\right) P^{-\frac{1}{2}} z$$
$$-2f(a, 0) z^T P^{\frac{1}{2}} BB^T P^{\frac{1}{2}} z, \quad z(t) \in \mathcal{B}_\Delta, \quad (37)$$

Using Lemma 1 we have that $2f(a, 0) \geq 1$, and we get

$$\dot{V} \leq z^T P^{-\frac{1}{2}} \left(A^T P + PA\right) P^{-\frac{1}{2}} z$$
$$-z^T P^{\frac{1}{2}} BB^T P^{\frac{1}{2}} z, \quad z(t) \in \mathcal{B}_\Delta, \quad (38)$$

and it follows that

$$\dot{V} \leq -z^T P^{-\frac{1}{2}} Q(\gamma) P^{-\frac{1}{2}} z, \quad z(t) \in \mathcal{B}_\Delta. \quad (39)$$

Since $\|z(t)\|$ is strictly decreasing, it is clear that $z(t_0) \in \mathcal{B}_\Delta$ implies that $z(t) \in \mathcal{B}_\Delta$, $t \geq t_0$. Moreover, since $x(t_0) \in \mathcal{X}$ implies that $z(t_0) \in \mathcal{B}_\Delta$ then (32) and (39) give

$$\dot{V} \leq -x^T Q(\gamma) x, \quad x(t_0) \in \mathcal{X}. \quad (40)$$

Hence, asymptotic stability follows from (35) and (40) and completes the proof. □

Remark 3 The high-gain result stated in this lemma is motivated from. and similar to, the low-and-high gain controller presented in [6]. In fact, taking $\kappa = 1 + \rho$, $\rho \geq 0$, we get the low-and-high gain controller. However, the proofs of stability differs and in this paper it is not required that the low-gain controller respects the input constraints. △

4 Main Results

We are now ready to state the main results, and still consider the system (1). First, we give sufficient condition for the existence of a linear control law stabilizing the system.

Theorem 1 Consider the system (1), assume that $w(t) = v(t) = 0$ and let $A \in \mathcal{M}_q^{n \times n}$ for some $q \in [0, n]$. Then, for any $\Delta > 0$, there exists a non-empty ball about the origin, $\mathcal{B}_2$, such that for any set of initial conditions satisfying $\mathcal{X}_2 \subseteq \mathcal{B}_2$, the system is locally asymptotically stabilizable by linear control laws. Moreover, if $\mathcal{X}_2 \subseteq \mathcal{B}_2$ then for any arbitrary large and bounded set of initial conditions $\mathcal{X}_1$ there exists linear control laws such that $\mathcal{X}_1 \times \mathcal{X}_2$ is contained in the region of attraction of the closed-loop system.

Proof: Let $P(\gamma)$ be the solution of (17) and consider the change of coordinates

$$z = P^{\frac{1}{2}}(\gamma) x, \quad (41)$$

where $P(\gamma) = P^{\frac{1}{2}}(\gamma) P^{\frac{1}{2}}(\gamma)$. Define a constant

$$\kappa \triangleq \|B_2^T \overline{P}_{22}^{\frac{1}{2}}\| \quad (42)$$

where $\overline{P}_{22}$ is the solution of (21) and let $\mathcal{B}_2$ be given by

$$\mathcal{B}_2 \triangleq \left\{ x_2 \in \mathbb{R}^{n-q} \; : \; \|\overline{P}_{22}^{\frac{1}{2}} x_2\| \leq 2\frac{\Delta}{\kappa} - \epsilon \right\}. \quad (43)$$

for some $\epsilon > 0$. Clearly, for any $\overline{P}_{22}$ and any $\Delta > 0$ there exists an $\epsilon > 0$ such that the ball $\mathcal{B}_2$ is non-empty.

Next, we need to show that for any $\mathcal{X}_2 \subseteq \mathcal{B}_2$ and for any arbitrary large and bounded $\mathcal{X}_1$ there exists a linear stabilizing control law such that $\mathcal{X}_1 \times \mathcal{X}_2$ is included in the region of attraction. The state transformation (41) can be written

$$z_1 = H_{11}(\gamma) x_1 + H_{12}(\gamma) x_2, \quad (44)$$
$$z_2 = H_{21}(\gamma) x_1 + H_{22}(\gamma) x_2 \quad (45)$$

with obvious definitions of the matrices H_{ij}, $i, j = 1, 2$. Then if $(z_1(t_0), z_2(t_0)) \in \mathcal{Z}_1(\gamma) \times \mathcal{Z}_2(\gamma)$, it follows that

$$\lim_{\gamma \to 0} \mathcal{Z}_1(\gamma) \times \mathcal{Z}_2(\gamma) = 0 \times \mathcal{Z}_2(0), \quad (46)$$

where

$$\mathcal{Z}_2(0) = \left\{ z_2 \in \mathbb{R}^{n-q} \; : \; \|z_2\| \leq 2\frac{\Delta}{\kappa} - \epsilon \right\}. \quad (47)$$

Then, using (43) it follows that there exists a γ^* such that

$$\|B^T P^{\frac{1}{2}}(\gamma)\| \, \|P^{\frac{1}{2}}(\gamma) x\| \leq 2\Delta. \quad (48)$$

Hence, we have verified that for any $\gamma \in (0, \gamma^*]$, the corresponding $P(\gamma)$ satisfies the conditions of Lemma 4 and the control law

$$u = -\kappa B^T P(\gamma) x \quad (49)$$

give asymptotic stability for any $\kappa \in [1, \infty)$. □

Remark 4 Notice that when $A \in \mathcal{M}_n^{n \times n}$, Theorem 1 states that the system is semi-globally stabilizable by linear state feedback. △

Theorem 1 implies that if the set $\mathcal{X}_2$ is small enough, depending on the saturation limit, Δ, it is possible to stabilize the unstable modes of the system. Moreover, if this is possible, then there exists a bounded controller stabilizing the system for *any* arbitrary large and bounded set $\mathcal{X}_1$.

Next, we give sufficient conditions for ultimately boundedness in the presence of disturbances. Let the measurement noise vector $v(t)$ be bounded, i.e. $v(t) \in \mathcal{V} \subset \mathbb{R}^n$, $t \geq t_0$ and writing $v(t) = [v_2^T(t), v_2^T(t)]^T$, then $v_2(t) \in \mathcal{V}_2 \subset \mathbb{R}^{(n-q)}$, $t \geq t_0$.

Theorem 2 Consider the system (1) with $A \in \mathcal{M}_q^{n \times n}$ for some $q \in [0, n]$ and let $\overline{P}_{22}$ be the solution of (21) and $s, \epsilon > 0$. If

$$\|B_2^T \overline{P}_{22}^{\frac{1}{2}}\| \, \|\overline{P}_{22}^{\frac{1}{2}} x_2(t_0)\| \leq \Delta + s - \epsilon, \quad (50)$$

for all $x_2(t_0) \in \mathcal{X}_2$ and

$$\sup_{t \geq t_0} \|B_2^T \overline{P}_{22} v_2(t)\| \leq \Delta - s - \epsilon, \quad (51)$$

for all $v_2(t) \in \mathcal{V}_2$, then there are sets $\mathcal{V}$ and $\mathcal{W}$ such that for $v(t) \in \mathcal{V}$, $t \geq t_0$, $w(t) \in \mathcal{W}$, $t \geq t_0$ and for any arbitrary large and bounded $\mathcal{X}_1$, there exists a linear control law such that the closed-loop system is ultimately bounded.

Proof: Consider the following Lyapunov function candidate

$$V = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} = \boldsymbol{z}^T\boldsymbol{z}, \tag{52}$$

where $\boldsymbol{z} = \boldsymbol{P}^{1/2}\boldsymbol{x}$ and $\boldsymbol{P} = \boldsymbol{P}(\gamma)$ for a fixed $\gamma \in (0,1]$. Let γ^* be such that for all $\gamma \in (0,\gamma^*]$,

$$\|\boldsymbol{B}^T\boldsymbol{P}^{\frac{1}{2}}\|\|\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{x}\| + \sup_{t\geq t_0}\|\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{v}\| \leq 2\Delta, \tag{53}$$

$\forall \boldsymbol{x}(t_0) \in \mathcal{X}$, $\boldsymbol{v}(t) \in \mathcal{V}$. Existence of γ^* follows from (50) and (51).

The time derivative of (52) becomes

$$\begin{aligned}\dot{V} &= \boldsymbol{z}^T\boldsymbol{P}^{-\frac{1}{2}}\left(\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{z}\\ &-2\boldsymbol{z}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{B}\sigma\left(\boldsymbol{B}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}+\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{v}\right)\\ &+\boldsymbol{w}^T\boldsymbol{E}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}+\boldsymbol{z}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{E}\boldsymbol{w},\end{aligned} \tag{54}$$

If we define two vectors

$$\boldsymbol{a} \triangleq \boldsymbol{B}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z},\ \ \boldsymbol{b} \triangleq \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{v} \tag{55}$$

it follows from (50) and (51) that $\boldsymbol{a} \in \mathcal{Y}_\Delta^+$, $\boldsymbol{b} \in \mathcal{Y}_\Delta^-$. Then, by Lemma 1,

$$\begin{aligned}\dot{V} &\leq \boldsymbol{z}^T\boldsymbol{P}^{-\frac{1}{2}}\left(\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{z}\\ &-\boldsymbol{z}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}-\boldsymbol{z}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{v}\\ &+\boldsymbol{w}^T\boldsymbol{E}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}+\boldsymbol{z}^T\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{E}\boldsymbol{w},\end{aligned} \tag{56}$$

for all $\forall \boldsymbol{z}(t) \in \mathcal{B}_\Delta$, where $\mathcal{B}_\Delta$ is given by (34). Finally, by defining

$$\begin{aligned}&l_1(\gamma) = \|\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P}\|,\ \ l_2(\gamma) = 2\|\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{E}\|,\\ &l_3(\gamma) = \|\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{Q}\boldsymbol{P}^{-\frac{1}{2}}\|\end{aligned} \tag{57}$$

we get

$$\dot{V} \leq -l_3(\gamma)\|\boldsymbol{z}\|\left(\|\boldsymbol{z}\| - \frac{l_1(\gamma)\|\boldsymbol{v}\| + l_2(\gamma)\|\boldsymbol{w}\|}{l_3(\gamma)}\right). \tag{58}$$

Hence, if

$$\|\boldsymbol{z}(t_0)\| \geq \frac{l_1(\gamma)\sup_{t\geq t_0}\|\boldsymbol{v}\| + l_2(\gamma)\sup_{t\geq t_0}\|\boldsymbol{w}\|}{l_3(\gamma)} \tag{59}$$

for all $\boldsymbol{z}(t_0) \in \partial\mathcal{B}_\Delta$, ultimately boundedness follows, and $\boldsymbol{z}(t)$ will never leave the set $\mathcal{B}_\Delta$. □

5 Illustrative Example

To illustrate the results of this paper, the benchmark example of balancing an inverted pendulum is considered. This system is open-loop exponentially unstable, and it is assumed to have limited control force. The state vector is defined to be

$$\boldsymbol{z} \triangleq [\ s,\ \dot{s},\ \theta,\ \dot{\theta}\ ]^T, \tag{60}$$

where s (m) is the cart displacement, $\dot{s}$ (m/s) is the linear velocity, θ (rad) is the pendulum angle and $\dot{\theta}$ (rad/s) is the angular velocity, see Figure 1.

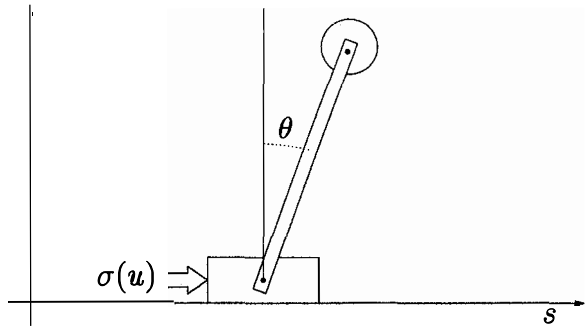

Figure 1: Sketch of the inverted pendulum.

The initial conditions of the system is assumed to satisfy

$$\begin{aligned}&\theta(0) \in [-0.17, 0.17],\quad \dot{\theta}(0) \in [-0.17, 0.17],\\ &s(0) \in [-10, 10],\quad\quad \dot{s}(0) \in [-1, 1].\end{aligned} \tag{61}$$

Consider the linearized model presented in [4],

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m+M)g}{Ml} & 0 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}\sigma(u) \tag{62}$$

where u (N) is the control force and $\sigma(u)$ is a saturation function given by

$$\sigma(u) = \text{sign}(u)\min\{|u|, \Delta\} \tag{63}$$

with $\Delta = 5$. The parameter values are taken as $m = M = 0.5$ (kg), $l = 1.4$ (m) and $g = 9.8$ (m/s^2) and results in open-loop eigenvalues of value 0, 0, 3.78 and −3.78.

The task is to derive a controller taking the pendulum to the origin. It is well-known that this is impossible to do globally when the input is bounded. This is a result of the system being open-loop unstable. In fact, it is impossible to semi-globally stabilize, and we want to do the best possible, which is to obtain local stability with $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$ contained in the region of attraction for any arbitrary large $\mathcal{Z}_1$.

Using the state transformation

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}' \end{bmatrix},\ \ \boldsymbol{T}' = \begin{bmatrix} -0.26 & 0.26 \\ 0.97 & 0.97 \end{bmatrix} \tag{64}$$

give the system dynamics

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2.56 & -2.56 \\ 0 & 0 & -3.78 & 0 \\ 0 & 0 & 0 & 3.78 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \\ -0.74 \\ -0.74 \end{bmatrix}\sigma(u) \tag{65}$$

where

$$\boldsymbol{x}_1 = [\ s\ \ \dot{s}\ \ -(1.95\,\theta - 0.52\,\dot{\theta})\ ], \tag{66}$$

$$\boldsymbol{x}_2 = 1.95\,\theta + 0.52\,\dot{\theta}. \tag{67}$$

Notice that in (65) we have $\boldsymbol{A} \in \mathcal{M}_3^{4\times 4}$. Since $\boldsymbol{A}_{22} = 3.78$ is a scalar, the ARE (21) is a scalar equation with solution

$$\overline{P}_{22} = 13.81. \tag{68}$$

In the new coordinates, we have $x_2(0) \in [-0.42, 0.42]$, and it follows that

$$\kappa = 0.74|\overline{P}_{22}^{\frac{1}{2}}|\,|\overline{P}_{22}^{\frac{1}{2}}x_2(0)| = 4.29 < 2\Delta. \quad (69)$$

Hence, the conditions of Lemma 2 are satisfied and there exist a linear controller stabilizing the system.

It can be shown that for $Q(\gamma) = \gamma I$, we get $\gamma^* = 0.067$. Taking $\gamma = 0.06 \in (0, \gamma^*]$, we get the control law

$$u = [\ -0.06 \quad -0.38 \quad -22.0 \quad -6.08\ ]\, z \quad (70)$$

A simulation with $z(0) = [10, 1, 0.17, 0.17]^T$ as initial condition. In Figure 2 the states converging to zero are shown in the upper left and upper right plots, while the input is shown in the lower plot.

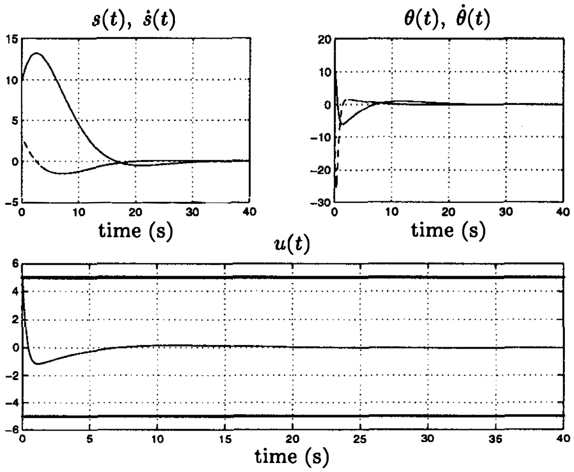

Figure 2: The time response of the states with initial condition $z(0) = [10, 1, 0.17, 0.17]^T$.

This example clearly illustrates the main result of this paper; If there are enough control authority to stabilize the $(\theta, \dot{\theta})$-dynamics, then no matter how far the pendulum is from $s = 0$ or with what speed it is traveling, there exists a linear control law bringing it back to $s = \dot{s} = 0$.

6 Conclusions

In this paper we have shown that for a class of linear unstable systems it is possible to obtain an arbitrary large and bounded region of attraction in certain directions of the state space. This result is stronger than local stability since the set of initial conditions is not required to be upper bounded. On the other hand it is a weaker result than semi-global stability, since the set of initial conditions cannot be arbitrary large in all directions of the state space. Moreover, we have given sufficient conditions for the existence of stabilizing controllers when the system is subject to plant disturbances and measurement noise.

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