FINITE ELEMENT MODELLING OF MOORING LINES

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Abstract In this paper, we develop a new finite element model for a cable suspended in water. Global existence and uniqueness of solutions of the truncated system is shown for a slightly simplified equation describing the motion of a cable having negligible added mass and supported by fixed end-points. Based on this, along with well known results on local existence and uniqueness of solutions for symmetrizable hyperbolic systems, we conjecture a global result for the initial-boundary value problem.

1 Introduction

This paper is a subset of another paper [1] dealing with position mooring systems (PM) for offshore oil production. PM systems have been commercially available since the late 1980's, and have proven to be a cost-effective alternative to permanent platforms for offshore oil production. In traditional testing of the performance of PM systems by means of computer simulations, tabulated static solutions of the cable equation have been coupled to the vessel dynamics. This approach is adequate for shallow waters. However, in deeper waters, dynamic interactions between the vessel and mooring system renders such a quasi-static approach inaccurate [3].

Software packages that solve the cable equation by means of the finite element method (FEM) are readily available. However, such general purpose FEM packages are not suited for control system design, and are usually slower than software tailored for a particular application. Moreover, the theoretical aspects, such as existence and uniqueness of solutions, are often taken for granted. In fact, FEM tools were developed and used, for instance in structural engineering, decades before a sound theoretical foundation was established [4].

In this paper, a new finite element model of a cable suspended in water is derived. The hydrodynamic loads on the cable are modelled according to *Morison's equation* (see for instance, [2]). For a slightly simplified equation, describing the motion of a cable having negligible added mass and supported by two fixed end-points, we show global existence and uniqueness of solutions of the truncated system, and conjecture a global result for the initial-boundary value problem.

2 PDE for the cable dynamics

The equation of motion of a cable with negligible bending and torsional stiffness is given by (see for instance [6])

$$\rho_0 \frac{\partial \vec{\mathbf{v}}(t,s)}{\partial t} = \frac{\partial}{\partial s}(T(t,s)\vec{\mathbf{t}}(t,s)) + \vec{\mathbf{f}}(t,s)(1+e(t,s))$$

where t is the time variable, and $s \in [0,L]$, $\vec{\mathbf{v}} :[t_0,\infty) \times [0,L] \to \mathbb{R}^3$ and $\vec{\mathbf{t}} :[t_0,\infty) \times [0,L] \to \mathbb{R}^3$ are distance along the unstretched cable, velocity and tangential vector, respectively. L is the length of the unstretched cable, ρ_0 is mass per unit length of unstretched cable, $T : [t_0,\infty) \times [0,L] \to \mathbb{R}$ is tension, $e : [t_0,\infty) \times [0,L] \to \mathbb{R}$ is strain and $\vec{\mathbf{f}} : [t_0,\infty) \times [0,L] \to \mathbb{R}^3$ is the sum of external forces (per unit length of unstreched cable) acting on the cable. By introducing the position vector $\vec{\mathbf{r}} :[t_0,\infty) \times [0,L] \to \mathbb{R}^3$, we get $\vec{\mathbf{t}} = \frac{1}{1+e}\frac{\partial \vec{\mathbf{r}}}{\partial s}$ such that

$$\rho_0 \frac{\partial^2 \vec{\mathbf{r}}}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial \vec{\mathbf{r}}}{\partial s}\right) + \vec{\mathbf{f}}(1+e)$$

Applying *Hooke's law* yields

$$\rho_0 \frac{\partial^2 \vec{\mathbf{r}}}{\partial t^2} = \frac{\partial}{\partial s}\left(EA_0 \frac{e}{1+e}\frac{\partial \vec{\mathbf{r}}}{\partial s}\right) + \vec{\mathbf{f}}(1+e)$$

where E is *Young's modulus* and A_0 is the cross-sectional area of the unstretched cable.

2.1 External forces

In addition to gravity, a submerged cable is subject to hydrostatic and hydrodynamic forces, i.e.

$$\vec{\mathbf{f}} = \vec{\mathbf{f}}_{(hg)} + \vec{\mathbf{f}}_{(dt)} + \vec{\mathbf{f}}_{(dn)} + \vec{\mathbf{f}}_{(mn)}$$

where $\vec{\mathbf{f}}_{(hg)}$ constitutes the bouyancy (gravity and hydrostatic) force per unit length of unstretched cable, $\vec{\mathbf{f}}_{(dt)}$ and $\vec{\mathbf{f}}_{(dn)}$ are tangential and normal hydrodynamic drag, respectively, per unit length of unstretched cable and $\vec{\mathbf{f}}_{(mn)}$ is the hydrodynamic inertia force per unit length of unstretched cable.

Gravity and hydrostatic forces

It is assumed that we can regard each element of the cable as completely surrounded by water so that

$$\vec{\mathbf{f}}_{(hg)} = \rho_0 \frac{\rho_c - \rho_w}{(1+e)\rho_c}\vec{\mathbf{g}}$$

where $\vec{\mathbf{g}} \in \mathbb{R}^3$ is the gravitational acceleration, ρ_c is density of the cable and ρ_w is density of the ambient water.

Hydrodynamic forces

From *Morison's equation,* see for instance [2], we get the following expression for hydrodynamic drag per unit length of unstretched cable

$$\begin{aligned}
\vec{\mathbf{f}}_{(dt)} &= -\frac{1}{2}C_{DT} d\rho_w \left|\vec{\mathbf{v}}\cdot\vec{\mathbf{t}}\right| (\vec{\mathbf{v}}\cdot\vec{\mathbf{t}})\vec{\mathbf{t}} = -\frac{1}{2}C_{DT} d\rho_w \left|\vec{\mathbf{v}}_t\right| \vec{\mathbf{v}}_t \\
\vec{\mathbf{f}}_{(dn)} &= -\frac{1}{2}C_{DN} d\rho_w \left|\vec{\mathbf{v}} - (\vec{\mathbf{v}}\cdot\vec{\mathbf{t}})\vec{\mathbf{t}}\right| (\vec{\mathbf{v}} - (\vec{\mathbf{v}}\cdot\vec{\mathbf{t}})\vec{\mathbf{t}}) = -\frac{1}{2}C_{DN} d\rho_w \left|\vec{\mathbf{v}}_n\right| \vec{\mathbf{v}}_n
\end{aligned}$$

where C_{DT} and C_{DN} are tangential and normal drag coefficients for the cable, respectively, and d is the cable diameter. The hydrodynamic inertia force per unit length of unstretched cable is given by:

$$\vec{\mathbf{f}}_{(mn)} = -C_{MN}\frac{\pi d^2}{4}\rho_w(\vec{\mathbf{a}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{t}})\vec{\mathbf{t}}) = -C_{MN}\frac{\pi d^2}{4}\rho_w \vec{\mathbf{a}}_n$$

where C_{MN} is a hydrodynamic mass coefficient and $\vec{\mathbf{a}}:[t_0,\infty)\times[0,L]\to\mathbb{R}^3$ is the acceleration. The subscripts n and t on $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ denote decompositions into the normal and tangential directions, respectively.

Formulation of the initial-boundary value problem

We have the following initial-boundary value problem

$$\rho_0 \frac{\partial^2 \vec{\mathbf{r}}}{\partial t^2} - \frac{\partial}{\partial s}\left(EA_0 \frac{e}{1+e}\frac{\partial \vec{\mathbf{r}}}{\partial s}\right) - (1+e)\left(\vec{\mathbf{f}}_{(hg)} + \vec{\mathbf{f}}_{(dt)} + \vec{\mathbf{f}}_{(dn)} + \vec{\mathbf{f}}_{(mn)}\right) = 0 \quad (1)$$

with boundary conditions

$$\vec{\mathbf{r}}(t,0) = \vec{\mathbf{r}}_0(0), \quad \vec{\mathbf{r}}(t,L) = \vec{\mathbf{r}}_0(L), \quad \text{for all } t \geq t_0$$

and initial conditions

$$\vec{\mathbf{r}}(t_0,s) = \vec{\mathbf{r}}_0(s), \quad \vec{\mathbf{v}}(t_0,s) = \vec{\mathbf{v}}_0(s)$$

Here, $\vec{\mathbf{r}}_0 : [0,L] \to \mathbb{R}^3$ and $\vec{\mathbf{v}}_0 : [0,L] \to \mathbb{R}^3$ are initial cable configuration and initial cable velocity, respectively.

3 Discretization into finite elements

Discretization of the initial-boundary value problem is performed using the Galerkin method and finite elements. This method consists of the following steps

1. The initial-boundary value problem (1) is transformed into the corresponding *generalized problem.* This is done by multiplying the equation by the functions $\vec{\mathbf{w}} \in \mathcal{V}$, and then integrating by parts over $[0, L]$. $\mathcal{V}$ is a suitable space of functions in which to search for a solution.
2. Restriction of $\vec{\mathbf{r}}$ and $\vec{\mathbf{w}}$ to appropriate finite-dimensional subspaces $\mathbb{V}_n \subset \mathcal{V}$, yields the Galerkin method.
3. Choosing the finite-dimensional subspaces such that they are spanned by bases consisting of so-called finite elements, yields a particularly simple set of ordinary differential equations. This is the finite element method.

The Galerkin equation resulting from (1) is given by

$$\begin{gathered}\frac{\rho_0 l}{6}(\ddot{\mathbf{r}}_{k-1} + 4\ddot{\mathbf{r}}_k + \ddot{\mathbf{r}}_{k+1}) + EA_0\left[\frac{e_k}{\varepsilon_k}\mathbf{l}_k - \frac{e_{k+1}}{\varepsilon_{k+1}}\mathbf{l}_{k+1}\right] = \\ \int_0^L \left(\vec{\mathbf{f}}_{(hg)} + \vec{\mathbf{f}}_{(dt)} + \vec{\mathbf{f}}_{(dn)} + \vec{\mathbf{f}}_{(mn)}\right)(1+e)\varphi_k ds \\ k = 1, 2, ..., n-1\end{gathered} \tag{2}$$

where

$$\begin{aligned}\mathbf{l}_k &= \mathbf{r}_k - \mathbf{r}_{k-1} \\ e_k &= \frac{1}{l}\left|\mathbf{r}_k - \mathbf{r}_{k-1}\right| - 1 = \frac{|\mathbf{l}_k|}{l} - 1 \\ \varepsilon_k &= l(1+e_k) = |\mathbf{l}_k| \\ \varphi_i(s) &= \begin{cases} 0 & s < (i-1)l \\ \frac{1}{l}s - (i-1) & (i-1)l \le s < il \\ -\frac{1}{l}s + i + 1 & il \le s < (i+1)l \\ 0 & (i+1)l \le s \end{cases}, \quad i = 0, 1, 2, ..., n\end{aligned}$$

The subscripts (hg), (dt), (dn) and (mn) stand for hydrostatic and gravity forces, tangential drag forces, normal drag forces, and hydrodynamic added inertia forces, respectively. n is the number of finite elements, and $l = L/n$ is the unstretched length of each element. The triangular form of the φ_i functions reflects the choice of a finite element basis for the subspaces $\mathbb{V}_n$. Note that in this form, algebraic expressions for the drag forces cannot be found. However, in Section 5, approximations are introduced that eliminate the need for numerical integration of these terms.

4 Existence and uniqueness of solutions

In this section we show existence and uniqueness of solutions for a slightly simplified equation under the assumption of strictly positive strain. This is the main contribution of the paper.

Assumption 1 *There exists a constant $c > 0$, such that*

i) $e(t, s) \ge c$ *for* $s \in [0, L]$ *and for all* $t \ge t_0$*, and;*
ii) $e_k(t) \ge c$ *for* $k = 1, 2, ..., n$ *and for all* $t \ge t_0$.

Neglecting the added mass term $\vec{\mathbf{f}}_{(mn)}$, which means that we assume drag dominant behaviour, and considering a damping term in the form

$$\vec{\mathbf{f}}_{(d)} = -\frac{1}{2}C_D d\rho_w \left|\vec{\mathbf{v}}\right| \vec{\mathbf{v}}$$

yield the following slightly modified initial-boundary value problem

$$\frac{\partial^2 \vec{\mathbf{r}}}{\partial t^2} - \frac{EA_0}{\rho_0}\frac{\partial}{\partial s}\left(\frac{e}{1+e}\frac{\partial \vec{\mathbf{r}}}{\partial s}\right) - \frac{\rho_c - \rho_w}{\rho_c}\vec{\mathbf{g}} + \frac{1}{2\rho_0}(1+e)C_D d\rho_w \left|\vec{\mathbf{v}}\right| \vec{\mathbf{v}} = 0 \tag{3}$$

with boundary conditions

$$\vec{\mathbf{r}}(t,0) = \vec{\mathbf{r}}_0(0), \quad \vec{\mathbf{r}}(t,L) = \vec{\mathbf{r}}_0(L), \quad \forall\ t \geq t_0$$

and initial conditions

$$\vec{\mathbf{r}}(t_0,s) = \vec{\mathbf{r}}_0(s), \quad \vec{\mathbf{v}}(t_0,s) = \vec{\mathbf{v}}_0(s)$$

Our goal is to apply Proposition 2.1 in [5, page 370], which states local existence and uniqueness of solutions for symmetrizable hyperbolic systems. Thus, we need to show that equation (3) is symmetrizable. Define $\mathbf{u}(t,s)$ as follows

$$\mathbf{u}(t,s) = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \triangleq \begin{bmatrix} \vec{\mathbf{r}} \\ \frac{\partial \vec{\mathbf{r}}}{\partial s} \\ \frac{\partial \vec{\mathbf{r}}}{\partial t} \end{bmatrix}$$

Carrying out the differentiation in the first term on the right hand side of (3), the equation, in terms of $\mathbf{u}$, can be written as

$$\mathbf{A}_0 \frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial s} + \mathbf{g} \tag{4}$$

where

$$\begin{aligned}
\mathbf{A}_0(t,s,\mathbf{u}) &= \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \frac{EA_0}{\rho_0}\left(\frac{\mathbf{u}_1\mathbf{u}_1^T}{(1+e)^3} + \frac{e}{1+e}\mathbf{I}\right) & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \\
\mathbf{A}_1(t,s,\mathbf{u}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{EA_0}{\rho_0}\left(\frac{\mathbf{u}_1\mathbf{u}_1^T}{(1+e)^3} + \frac{e}{1+e}\mathbf{I}\right) \\ 0 & \frac{EA_0}{\rho_0}\left(\frac{\mathbf{u}_1\mathbf{u}_1^T}{(1+e)^3} + \frac{e}{1+e}\mathbf{I}\right) & 0 \end{bmatrix} \\
\mathbf{g}(t,s,\mathbf{u}) &= \begin{bmatrix} \mathbf{u}_2 \\ 0 \\ \frac{\rho_c-\rho_w}{\rho_c}\vec{\mathbf{g}} - \frac{1}{2\rho_0}(1+e)C_D d\rho_w \left|\mathbf{u}_2\right| \mathbf{u}_2 \end{bmatrix}
\end{aligned}$$

In (4), we have multiplied the equation by the matrix $\mathbf{A}_0$, which is symmetric positive definite for $s \in [0,L]$ and for all $t \geq t_0$ under Assumption 1 (in fact, there exists a constant c, such that $\mathbf{A}_0 \geq c\mathbf{I} > 0,\ \ s \in [0,L],\ \ \forall t \geq t_0$). Notice that the matrix $\mathbf{A}_1$ is rendered symmetric, by means of the *symmetrizer* $\mathbf{A}_0$. Thus, Proposition 2.1 in [5, page 370], provides local existence of a unique solution to (4). However, based on the following arguments, we will conjecture that the solution can be continued for all time. For the Galerkin equation corresponding to (3), which is given by

$$\begin{aligned}
\frac{\rho_0 l}{6}(\ddot{\mathbf{r}}_{k-1} + 4\ddot{\mathbf{r}}_k + \ddot{\mathbf{r}}_{k+1}) + EA_0\left[\frac{e_k}{\varepsilon_k}\mathbf{l}_k - \frac{e_{k+1}}{\varepsilon_{k+1}}\mathbf{l}_{k+1}\right] &= \int_0^L \left(\vec{\mathbf{f}}_{(hg)} + \vec{\mathbf{f}}_{(d)}\right)(1+e)\varphi_k ds \\
k &= 1,2,...,n-1
\end{aligned} \tag{5}$$

we can state the following result (proven in [1]).

Theorem 1 *For any $n \in \{2,3,4,...\}$, let the initial state $(\mathbf{r}(t_0), \mathbf{v}(t_0)) = (\mathbf{r}_0, \mathbf{v}_0)$ be given. If Assumption 1 holds, then there exists a unique solution of (5) for all $t \geq t_0$.*

Theorem 1 implies that $\mathbf{u}_n \in H^k([0,L])$, for $t \geq t_0$, where $H^k([0,L])$ denotes the Sobolev space defined by

$$H^k([0,L]) = \left\{ \mathbf{u} \in L^{\mathbf{2}}([0,L]) \;\middle|\; \frac{\partial^l \mathbf{u}}{\partial s^l} \in L^{\mathbf{2}}([0,L]), \;\; 0 < l \leq k \right\}$$

with the natural norm

$$\|\mathbf{u}\|_{H^k([0,L])} = \sum_{l=0}^{l=k} \left\| \frac{\partial^l \mathbf{u}}{\partial s^l} \right\|_{L^{\mathbf{2}}([0,L])}$$

and $\mathbf{u}_n$ is given in terms of the finite element basis defined in Section 3, that is

$$\mathbf{u}_n(t,s) = \sum_{i=0}^{n} \begin{bmatrix} \mathbf{r}_i(t)\varphi_i(s) \\ \mathbf{r}_i(t)\frac{\partial \varphi_i(s)}{\partial s} \\ \mathbf{v}_i(t)\varphi_i(s) \end{bmatrix} \tag{6}$$

In fact, Theorem 1 implies that there exists a constant c, independent of k and n, such that

$$\|\mathbf{u}_n\|_{H^k([0,L])} \leq c, \;\; \forall t \geq t_0, \;\; n = 2, 3, ...$$

Based on the above considerations, along with the results of Chapter 16, Sections 1 and 2 in [5, pages 359-372], we conjecture the following.

Conjecture 1 *Suppose Assumption 1 holds, and that* $\mathbf{u}(0,s) \in H^k([0,L])$, *with* $k \geq 2$. *Then there exists a unique solution* $\mathbf{u} \in C([t_0,\infty), H^k([0,L]))$, *to the initial-boundary value problem (4). Moreover, the sequence of solutions* $\mathbf{u}_n$ *(as given in (6)) of the Galerkin equation (5), converges to* $\mathbf{u}$ *in the following sense*

$$\|\mathbf{u} - \mathbf{u}_n\|_{H^k([0,L])} \to 0 \;\; \textit{as} \; n \to \infty$$

Remark 2 *We stress the fact that since Theorem 1 and Conjecture 1 are stated under Assumption 1, global solutions are not guaranteed for all initial conditions. The problem of finding conditions on the initial data under which Assumption 1 holds (for all* $t \geq t_0$*), is outside the scope of this work.*

5 Implementation

It is desirable to apply certain approximations to the terms of equation (2) in order to simplify implementation. Looking at the k^{th} node, we see by inspection of equation (2), that it takes an advantageous form if the following approximations are applied

$$\begin{aligned} \dot{\mathbf{r}}_{k-1} &\approx \dot{\mathbf{r}}_k, \;\; \dot{\mathbf{r}}_{k+1} \approx \dot{\mathbf{r}}_k \\ \ddot{\mathbf{r}}_{k-1} &\approx \ddot{\mathbf{r}}_k, \;\; \ddot{\mathbf{r}}_{k+1} \approx \ddot{\mathbf{r}}_k \end{aligned}$$

With these approximations, equation (2) reduces to the following:

$$\begin{gathered} \left[\left(\rho_0 l + \frac{C_1}{2}(\varepsilon_k + \varepsilon_{k+1}) \right) \mathbf{I}_{3\times3} - \frac{C_1}{2} \left(\frac{\mathbf{l}_k \mathbf{l}_k^T}{\varepsilon_k} + \frac{\mathbf{l}_{k+1} \mathbf{l}_{k+1}^T}{\varepsilon_{k+1}} \right) \right] \ddot{\mathbf{r}}_k = \\ \mathbf{f}_{k_{(hg)}} + \mathbf{f}_{k_{(dt)}} + \mathbf{f}_{k_{(dn)}} + \mathbf{f}_{k_{(r)}}, \;\; k = 1, 2, ..., n-1 \end{gathered} \tag{7}$$

where

$$
\begin{aligned}
\mathbf{f}_{k_{(r)}} &= EA_0 \left[\frac{e_{k+1}}{\varepsilon_{k+1}} \mathbf{l}_{k+1} - \frac{e_k}{\varepsilon_k} \mathbf{l}_k \right] \\
\mathbf{f}_{k_{(hg)}} &= l\rho_0 \frac{\rho_c - \rho_w}{\rho_c} \left[\begin{array}{ccc} 0 & 0 & g \end{array} \right]^T \\
\mathbf{f}_{k_{(dt)}} &= -\frac{C_2}{2} \left[|\dot{\mathbf{r}}_k \cdot \mathbf{l}_k| \frac{\mathbf{l}_k \mathbf{l}_k^T}{\varepsilon_k^2} + |\dot{\mathbf{r}}_k \cdot \mathbf{l}_{k+1}| \frac{\mathbf{l}_{k+1} \mathbf{l}_{k+1}^T}{\varepsilon_{k+1}^2} \right] \dot{\mathbf{r}}_k \\
\mathbf{f}_{k_{(dn)}} &= -\frac{C_3}{2} \left[\varepsilon_k \left| \left(\mathbf{I}_{3\times 3} - \frac{\mathbf{l}_k \mathbf{l}_k^T}{\varepsilon_k^2} \right) \dot{\mathbf{r}}_k \right| \left(\mathbf{I}_{3\times 3} - \frac{\mathbf{l}_k \mathbf{l}_k^T}{\varepsilon_k^2} \right) \right. \\
&\quad \left. + \varepsilon_{k+1} \left| \left(\mathbf{I}_{3\times 3} - \frac{\mathbf{l}_{k+1} \mathbf{l}_{k+1}^T}{\varepsilon_{k+1}^2} \right) \dot{\mathbf{r}}_k \right| \left(\mathbf{I}_{3\times 3} - \frac{\mathbf{l}_{k+1} \mathbf{l}_{k+1}^T}{\varepsilon_{k+1}^2} \right) \right] \dot{\mathbf{r}}_k \\
C_1 &= C_{MN} \frac{\pi d^2}{4} \rho_w, \;\; C_2 = \frac{1}{2} C_{DT} d\rho_w, \;\; C_3 = \frac{1}{2} C_{DN} d\rho_w
\end{aligned}
$$

$\mathbf{I}_{3\times 3}$ is the 3×3 identity matrix, and the subscript (r) stands for internal reaction forces. Clearly, in the limit as $n \to \infty$, (2) and (7) are identical. Modelling a moored vessel is now a matter of assembling the above equations for each mooring line. The details of this procedure are available in [1].

6 Conclusions

In this paper, we have developed a new finite element model for a cable suspended in water. Global existence and uniqueness of solutions of the truncated system is shown for a slightly simplified equation describing the motion of a cable with negligible added mass and supported by fixed end-points. Based on this, along with well known results on local existence and uniqueness of solutions for symmetrizable hyperbolic systems, we conjecture a global result for the initial-boundary value problem.

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