Formation Control of Marine Craft using Constraint Functions

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***Abstract*— This article presents a method for formation controlof marine surface vessels inspired by Lagrangian mechanics. The desired formation configuration is given as a set of constraint functions. The functions are treated analytically and by using feedback from the imposed constraint functions, constraint forces arise. Since the constraint functions are designed for a desired effect, the forces can be seen as control laws. These forces act so that the constraint functions are satisfied in order to keep the formation assembled during operations. Examples of constraint functions that can be used to maintain a formation are presented. Simulations with surface vessel models have been performed to illustrate the proposed method.**

I. Introduction

The fields of coordination and formation control with applications towards mechanical systems, ships, aircraft, unmanned vehicles, spacecraft, etc., have been the object of recent research efforts in the last few years. This interest has gained momentum due to technological advances in the development of powerful control techniques for single vehicles, the increasing computation and communication capabilities, and the ability to create small, low-power and low-cost systems. Researchers have also been motivated by formation behaviors in nature, such as flocking and schooling, which benefits the animals in different ways [1], [2], [3], [4]. Many models from biology have been developed to give an understanding of the traffic rules that govern fish schools, bird flocks, and other animal groups, which again have provided motivation for control synthesis and computer graphics, see [5], [6] and the references therein.

The interest in cooperative behavior in biological systems, and the mixture with the control systems field, have led to observations and models that suggests that the motion of groups in nature applies a distributed control scheme. Further, the members of the formation are constrained by the position, orientation, and speed of their neighbors [1]. In the framework of [7] this has been translated into artificial potential functions, dependent on the relative distance between neighbors, that define the interaction between vehicles in the formation.

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There exists a large number of publications on the fields of cooperative and formation control – recent results can be found in [4], [8], [9], [10] and [11]. See the papers and references therein for an thorough overview. A short introduction is given here. There exists roughly three approaches to vehicle formation control in the literature: leader-following, behavioral methods and virtual structures.

Briefly explained, the leader-following architecture defines a leader in the formation while the other members of the formation follows that leader. Variations on this theme includes multiple leaders, forming a chain, and other tree topologies. The behavioral approach prescribes a set of desired behaviors for each member in the group, and weight them such that desirable group behavior emerges. Possible behaviors include trajectory and neighbor tracking, collision and obstacle avoidance, and formation keeping.

In the virtual structure approach, the entire formation is treated as a single, *virtual*, structure. Virtual structures have been achieved by for example, having all members of the formation tracking assigned nodes which move through space in the desired configuration, and by using formation feedback to prevent members leaving the formation [12]. In [13] each member of the formation tracks a virtual element, while the motion of the elements are governed by a formation function that specifies the desired geometry of the formation.

The main issue in this paper is the formation control aspect in cooperative behavior of marine craft. More specifically: how classic and powerful tools from analytical mechanics for multi-body dynamics [14] can be used for formation control issues. A collection of independent bodies/vehicles can be controlled as a formation by introducing functions that describe a vehicles behavior with respect to the others. By treating these functions as mechanical constraint functions in an analytical setting, stable control laws that maintain the structure of a formation emerge. In this way, the coordinated movement of the formation is decided by forces that maintain the constraints at all times.

Mechanical constraint forces, which cause the bodies to act in accordance with the constraints, are well known from the early days of analytical mechanics [15] and has been used with success, e.g. in computer graphics applications [16], [17]. This paper will show how constraint functions impose constraint forces which maintain the configuration of a formation as a virtual structure. The formation is also maintained when some, or all, of the members are exposed to external forces and disturbances. The same approach can be used with several, non-conflicting, constraint functions. The intersection of these functions defines the control objective for the total system–see Figure 1. Together, the constraints form control laws that both govern the movement of the entire formation and perform a specific task given by the imposed constraint function(s).

The rest of this paper is organized as follows. Section II presents a model for a formation of homogenous systems with constraints, methods for stabilizing constraints, and Section III shows how constraints can be imposed for control purposes. Section IV gives examples of formation control of marine craft with constraints. A comment regarding signal communication is given in Section V and Section VI contains some concluding remarks.

II. Modeling and Control

Consider n systems of degree of freedom m with kinetic and potential energy, $\mathcal{T}_i$ and $\mathcal{U}_i$, respectively. The Lagrangian of the total system is then

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \sum_{i=1}^{n} \mathcal{T}_i - \mathcal{U}_i.$$

Suppose there exists kinematic relations

$$\mathcal{C}(q) = 0,\ \mathcal{C}(q) : \mathbb{R}^{nm} \to \mathbb{R}^p \quad (1)$$

between the generalized coordinates which restrict the state space to a *constraint manifold* $\mathcal{M}_c$. We denote $\mathcal{C}(q)$ as the *constraint function*, where $q \in \mathbb{R}^{nm}$ contains the generalized positions, q_1,...,q_n. We know, from [14], that the forces that maintain the kinematic constraints adds potential energy to the system. This gives the following, modified, Lagrangian

$$\bar{\mathcal{L}} = \mathcal{T} - \mathcal{U} + \lambda^\top \mathcal{C}(q)$$

where $\lambda \in \mathbb{R}^p$ is the Lagrangian multiplier(s). To obtain the equations of motion, we apply the Euler-Lagrange differential equations with auxiliary conditions for $i = 1, \ldots, nm$,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial \mathcal{C}(q)}{\partial q_i}^\top + \frac{\partial \lambda}{\partial q_i}^\top \mathcal{C}(q) = \tau_i$$

which implies

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial \mathcal{C}(q)}{\partial q_i}^\top &= \tau_i \\ \mathcal{C}(q) &= 0 \end{aligned} \quad (2)$$

where τ_i is the generalized external force associated with coordinate q_i. This is actually a differential-algebraic equation (DAE) with index 2, which means that $\mathcal{C}(q) = 0$ must be differentiated twice before it can be converted into an ODE.

To explain why differentiaon is needed, a small explanation is in order. Equation (1) constrains the systems motion to a subset, $\mathcal{M}_c \subseteq \mathbb{R}^{2nm-p}$, of the state space where $\mathcal{C}(q) = 0$. Since we want to keep the systems on $\mathcal{M}_c$, neither the velocity nor the acceleration should violate the constraints. To find the velocities that satisfy the constraints, the kinematic admissible velocities, the constraint function is differentiated with respect to time. Similarly, we differentiate twice to find the kinematic admissible acceleration of the constraints. This gives the additional conditions

$$\begin{aligned} \dot{\mathcal{C}}(q) &= W(q)\dot{q} = 0 \\ \ddot{\mathcal{C}}(q) &= W(q)\ddot{q} + \dot{W}(q)\dot{q} = 0 \end{aligned} \quad (3)$$

where $W(q) \in \mathbb{R}^{mn \times p}$ is the Jacobian of the constraint function, i.e. $W(q) = \frac{\partial \mathcal{C}(q)}{\partial q}$. A combination of (2) and (3) yields an expression for the Lagrangian multiplier:

$$W M^{-1} W^\top \lambda = W M^{-1} \tau + \dot{W}\dot{q}$$

The expression for the constraint forces that maintain the constraints are found from (2):

$$\tau_{\text{constraint}} = -\frac{\partial \mathcal{C}(q)}{\partial q}^\top \lambda = -W(q)^\top \lambda.$$

Consider, for simplicity, a formation of n point masses with kinetical energy

$$\mathcal{T} = \frac{1}{2}\dot{q}^\top M \dot{q}$$

and where the only potential energy arise from the constraint function. This gives the equations of motion

$$M\ddot{q} + W(q)^\top \lambda = \tau \quad (4)$$

where τ is an external force.

Assumption A1: The Jacobian $W(q)$ has full row-rank, i.e., the constraints are not conflicting or redundant.

Note that redundant or conflicting constraints arise when one, or more, row in $\mathcal{C}$ is a linear combination of other rows, or when the functions are contradicting. An example would be the same constraint function appearing twice in $\mathcal{C}(q)$.

Assumption A1 guarantees that $WM^{-1}W$ exists since M is positive definite, hence M^{-1} exists and $WM^{-1}W^\top$ is nonsingular. Thus, the expression can be solved for λ and used in (4).

A. Stabilization

The objective of introducing constraint functions is to reduce the work space of the total system according to the given constraints, i.e., the different vehicles in a formation are not free to move at will but are restricted to the formation constraints. Consider the case where the constraint function $\mathcal{C}(q)$ consists of three constraints, as shown in Figure 1. With each $\mathcal{C}_i$ restricting the statespace to different subsets, the constraint manifold $\mathcal{M}_c$ is the resulting intersection where all the constraints are fulfilled. Hence, the control objective states that the system should evolve on this manifold for all times.

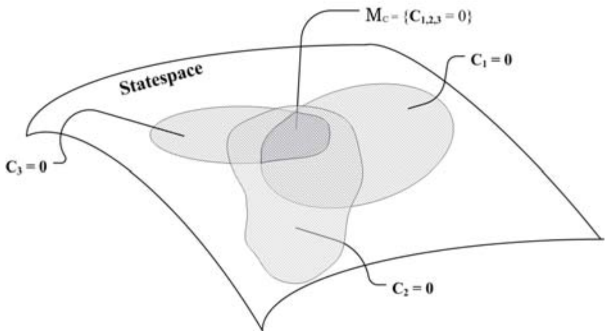

Fig. 1. Statespace and the constraint manifold: intersection of three constraints.

If the system starts on the constraint manifold $\mathcal{M}_c$, that is, the initial conditions $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$ satisfy $(q_0, \dot{q}_0) \in \mathcal{M}_c$ such that

$$\mathcal{C}(q_0) = 0 \text{ and } \dot{\mathcal{C}}(q_0, \dot{q}_0) = 0$$

and the force τ does not perturb the system so that $(q, \dot{q})$ leaves $\mathcal{M}_c$, then the system is well behaved and $(q, \dot{q}) \in \mathcal{M}_c$ for all times. However, if the initial conditions are not in $\mathcal{M}_c$, or the system is perturbed such that $(q, \dot{q}) \notin \mathcal{M}_c$, feedback must be introduced to stabilize the constraint function.

We want to investigate stability of the constraint, and in terms of set-stability we look at stability of the set

$$\mathcal{M}_c = \{(q, \dot{q}) : \mathcal{C}(q) = 0,\ W(q)\dot{q} = 0\}.$$

Consider the case when $\tau \neq 0$ in (4), and suppose that $(q_0, \dot{q}_0) \notin \mathcal{M}_c$. From the last section we have that

$$\ddot{\mathcal{C}}(q, \dot{q}) = 0. \tag{5}$$

Equation (5) is in fact unstable – in the case $\mathcal{C}(q)$ is a scalar function it can be seen as a transfer function with two poles at the origin. Hence, if $\mathcal{C}(q) = 0$ is not fulfilled initially, the solution might blow up in finite time. Even if $\mathcal{C}(q_0) = 0$, this might happen with noise present on the measurements of q and $\dot{q}$. This unstability is, in fact, an inherent property of higher-index DAEs [18], and is one of the reasons numerical methods for differential-algebraic equations has received special attention [19] in, e.g., modelling of mechanical systems, [20].

However, if we introduce feedback from the constraints in the expression for the Lagrangian multiplier,

$$WM^{-1}W^{\top}\lambda = WM^{-1}\tau + \dot{W}\dot{q} + K_d\dot{\mathcal{C}}(q, \dot{q}) + K_p\mathcal{C}(q) \tag{6}$$

where $K_p,\ K_d \in \mathbb{R}^{p\times p}$ and positive definite, we stabilize the constraint acceleration

$$\ddot{\mathcal{C}} = -K_d\dot{\mathcal{C}}(q, \dot{q}) - K_p\mathcal{C}(q). \tag{7}$$

By standard Lyapunov arguments, equation (7) guarantees that $(q, \dot{q}) \to \mathcal{M}_c$, i.e. the constraint functions are fulfilled.

III. Control with Constraint Functions

So far the focus has been on systems with constraints without discussing how the constraints arise. In the control literature, the main focus has been on models where constraints are inherently in the system as they are all based on how the model or the environment constrains the system. Some of the difficulties related to simulations of constrained systems and ways to solve them are described in [21].

However, if a control objective can be defined as one or more inter-vehicle constraint functions and if these *constraints are imposed* on the system, the framework for stabilization of constraints described earlier can be used to design control laws that force the system to behave in accordance with the control objective. The control laws which utilize feedback from the constraints are applicable for a wide variety of purposes, see [22] for a more thorough description and more examples.

A. Examples of Constraint Functions

To see how formation control can be achieved, this section will consider some examples of constraints that can be imposed to coordinate and control certain aspects of the group. To keep the notation compact, q_i and $\dot{q}_i = v_i$ are used for position and velocity of vehicle i, and collected into vector notation as $q = [q_1^{\top}, \ldots, q_n^{\top}]^{\top}$ and $v = [v_1^{\top}, \ldots, v_n^{\top}]^{\top}$.

1) Distance Between Members: To keep a fixed distance between members of the formation, functions arising from norms in mathematics can be used. To maintain a relative distance r_{ij} between group members i and j, let the function be defined by

$$\mathcal{C}_{rd}(q) = (q_i - q_j)^{\top}(q_i - q_j) - r_{ij} = 0. \tag{8}$$

If the control objective implies a stricter formation, with fixed offsets in the direction of each coordinate axis, consider the alternative distance function

$$\mathcal{C}_{fd}(q) = q_i - q_j - o_{ij} = 0 \tag{9}$$

where o_{ij} describes the offset between members i and j.

Furthermore, the entire formation structure can be decided by using a combination of distance constraint functions. For example, two vehicles with one $\mathcal{C}_{rd}$-function is a line-formation, three vehicles with three $\mathcal{C}_{rd}$-functions form a triangle, and so on. By using the $\mathcal{C}_{fd}$-functions, constraints can also be imposed on the orientation, and the desired offset between two members can be limited to a certain coordinate axis. The last approach can be utilized in a formation of AUVs moving on a horizontal plane.

An illustration of the two constraint functions is shown in Figure 2.

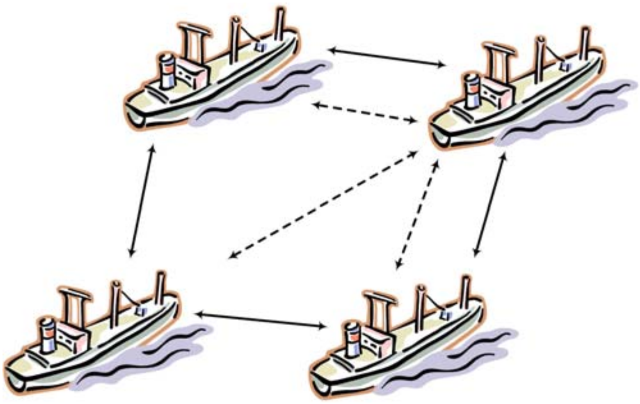

Fig. 2. Different constraint functions acting between vessels determine collective motion, $\mathcal{C}_{rd}$ (–) and $\mathcal{C}_{fd}$ (- -).

2) Time-varying functions: The constraint function need not only be dependent on generalized positions only. Instead, by letting the constraint functions depend on time, the formation can, among other things, change its configuration during operations according to a time-dependent signal. Consider the time-varying version of the relative-distance constraint function:

$$\mathcal{C}_{rdt}(q,t) = (q_i - q_j)^\top (q_i - q_j) - r_{ij}(t) = 0. \tag{10}$$

The addition of the time variable in the constraint function leads to a different expression for constraint velocity and acceleration, but the procedure to find the Lagrange multiplier is straightforward. Note that the expression for the constraint forces remains the same in this case, i.e., $\tau_{\text{constraint}} = -W^\top \lambda$.

3) Inequalities: Inequality constraint functions on the form $c(q) \geq 0$ can be treated within the framework presented in this paper through the logarithmic barrier function [23] of the form

$$\mathcal{C}_{ie}(q) = -\sum_i \log c_i(q) = 0. \tag{11}$$

This sort of function can be used when the control objective is to keep the members of the formation more than a certain distance away from each other, rather than at a fixed, desired, distance, while still keeping the formation assembled.

IV. Case Studies

A. Formation Assembling

We consider a formation of three vessels where the control objective is to assemble the craft in a predefined configuration, e.g., in order to be in position to tow a barge or another object. The purpose is to show that assembling of the individual vessels into a formation can be done by imposing constraints.

The equations of motion for a marine vessel in the body-fixed frame, derived analytically in [24] using an energy approach, are

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} + C(\nu)\dot{\nu} + D(\nu)\nu + g(\eta) = \tau$$

where $\eta = [x, y, \psi]^\top$ is the Earth-fixed position vector, (x, y) is the position on the ocean surface and ψ is the heading angle (yaw), and $\nu = [u, v, r]^\top$ is the body-fixed velocity vector. The model matrices $M = M^\top > 0$, C, and D denote inertia, Coriolis plus centrifugal and damping, respectively, while g is a vector of generalized gravitational forces and $R = R(\psi) \in SO(3)$ is the rotation matrix between the body and Earth coordinate frame. For more details regarding ship modeling, the reader is suggested to consult [24], [25], and [26]. The addition of the potential energy from the constraints gives

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} + C(\nu)\dot{\nu} + D(\nu)\nu + g(\eta) = \tau - J(\eta)^\top \lambda$$

These equations of motion can be transformed to the Earth-fixed frame by the kinematic transformation in [24, Ch. 3.3.1] which results in

$$M_\eta(\eta)\ddot{\eta} + n(\nu, \eta, \dot{\eta}) = \tau_\eta - R(\psi) J(\eta)^\top \lambda \tag{12}$$

where $n(\nu, \eta, \dot{\eta}) = C_\eta(\nu, \eta)\dot{\eta} + D_\eta(\nu, \eta)\dot{\eta} + g_\eta(\eta)$ and $\tau_\eta = R(\psi)\tau$.

Consider the constraint function

$$\mathcal{C}_1(\eta) = \begin{bmatrix} (\eta_1^* - \eta_2^*)^\top (\eta_1^* - \eta_2^*) - r_{12}^2 \\ (\eta_2^* - \eta_3^*)^\top (\eta_2^* - \eta_3^*) - r_{23}^2 \\ (\eta_3^* - \eta_1^*)^\top (\eta_3^* - \eta_1^*) - r_{31}^2 \end{bmatrix} = 0 \tag{13}$$

where $\eta_i^* \in \mathbb{R}^2$ is the reduced position vector without the orientation ψ and $r_{ij} \in \mathbb{R}$ is the distance between vessel i and j. This constraint enables us to specify the positions of each vessel with respect to the others, and the constraint manifold is now equivalent to the formation configuration, and by the previous sections we know that by using feedback from the constraints we can stabilize $\mathcal{M}_c$, and hence the stabilization of the constraint functions gives control laws for formation assembling. We assume there is no external force or control law acting on the formation, i.e. $\tau = 0$. From the ship model and the constraint, we have

$$M_\eta\ddot{\eta} + n(\nu, \eta, \dot{\eta}) = -R(\psi) W(\eta)^\top \lambda$$

where $M_\eta = M_\eta(\eta) = \operatorname{diag}(M_{\eta 1}, M_{\eta 2}, M_{\eta 3})$, $\eta = [\eta_1^\top, \eta_2^\top, \eta_3^\top]^\top$, and so on. Further, the Lagrangian multiplier can be obtained from

$$W M_\eta^{-1} R W^\top \lambda =$$
$$- W M_\eta^{-1} n(\nu, \eta, \dot{\eta}) + \dot{W}(\eta)\dot{\eta} + K_d \dot{\mathcal{C}}(\eta) + K_p \mathcal{C}(\eta).$$

M_η is a block-diagonal matrix consisting of positive definite matrices, which makes M_η positive definite. Further, since R is always non-singular and W has full rank due to Assumption A1, $W M_\eta^{-1} R W^\top$ is invertible.

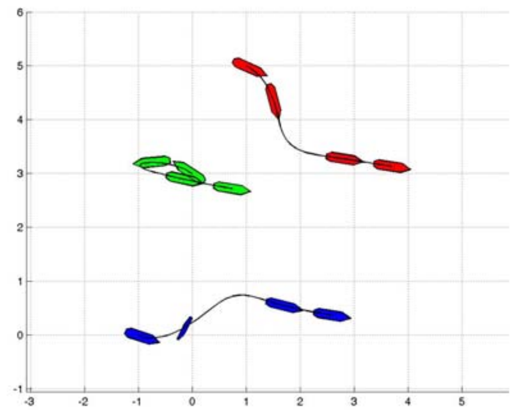

Fig. 3. Position of vessels during assembling. Red indicates vessel 1, green vessel 2, and blue vessel 3.

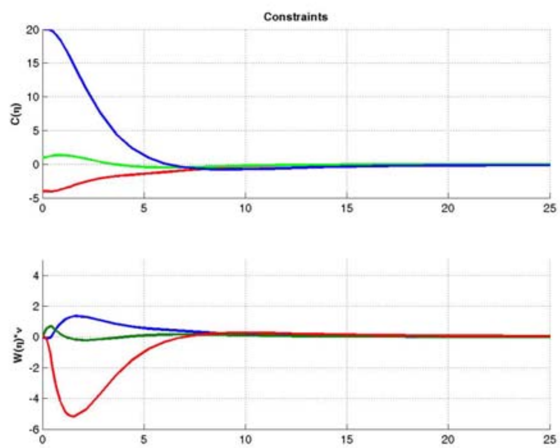

Fig. 4. Response of Assembling constraints.

The control parameters chosen to stabilize the constraint function $\mathcal{C}_2$ are $K_p = 1.6I_3$, $K_d = 0.64I_3$, the formation is defined by $r_{12} = 2$, $r_{23} = 3$, $r_{31} = 3$, and the desired position for the first vessel is $\eta_{\text{des}} = [2,4,0]^\top$. The vessels start in $\eta_{10} = [1,5,0]^\top$, $\eta_{20} = [0,3,0]$, and $\eta_{30} = [-1,0,0]$ – all with zero initial velocity.

The time-plots of the constraint function $\mathcal{C}_1$ and its time-derivative, $W_1(\eta)\dot{\eta}$, are shown in Figure 4. The constraints and velocity terms converge to zero, and the constraint manifold is reached. The vessels have converged to the nearest positions where the constraints are fulfilled, and the formation is hence assembled in the desired configuration.

B. Change of Configuration

In this section we will investigate how the formation can change its configuration during operations. Consider the case in IV-A, but let the desired distance between each

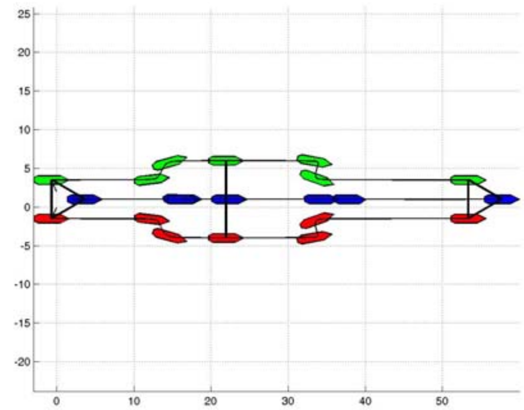

Fig. 5. Formation changes from triangular- to line- and back to triangular formation again.

member be time-dependent

$$\mathcal{C}_1(\eta,t) = \begin{bmatrix} (\eta_1^* - \eta_2^*)^\top (\eta_1^* - \eta_2^*) - r_{12}(t)^2 \\ (\eta_2^* - \eta_3^*)^\top (\eta_2^* - \eta_3^*) - r_{23}(t)^2 \\ (\eta_3^* - \eta_1^*)^\top (\eta_3^* - \eta_1^*) - r_{31}(t)^2 \end{bmatrix} = 0.$$

The objective in this simulation is to modify the formation configuration from a triangular shape to a line and then back to a triangular formation again. This is done by keeping r_{23} and r_{31} constant at 5 and let r_{12} be 5 (triangular shape) or 10 (line formation). To ensure that the $r_{12}(t)$ is smooth enough, the $\arctan(\cdot)$ function is used for the transitions. A step-function in $r_{12}(t)$ is not desirable since its derivative is not continuous.

The Lagrangian multiplier can be obtained from

$$\begin{aligned} WM_\eta^{-1}RW^\top\lambda &= WM_\eta^{-1}(\tau_\eta - n(\nu,\eta,\dot{\eta})) \\ &\quad + \dot{W}(\eta)\dot{\eta} + \frac{\partial^2\mathcal{C}}{\partial t^2} + K_d\dot{\mathcal{C}}(\eta,t) + K_p\mathcal{C}(\eta,t)\,. \end{aligned}$$

where τ_η consists of an external force on one of the vessels in the direction shown in Figure 5.

The vessels start with zero velocity at their initial positions, and the control gains are chosen as $K_p = 9$ and $K_d = 6$. The position of the vessels at different times is shown in Figure 5, while the time-evolution of the constraints are shown in Figure 6. The members of the formation assembles initially into a triangular formation. After 150 s the formation changes into a line formation where the members travel side by side, and, finally, after 350 s, they assemble into a triangular formation again. The changes in $r_{12}(t)$ can be seen in the time-response of the constraint functions.

V. Signal Communication Requirements

The control laws that follow from the design requires information about the position and velocities of neighboring vehicles, and are designed and implemented as a separate

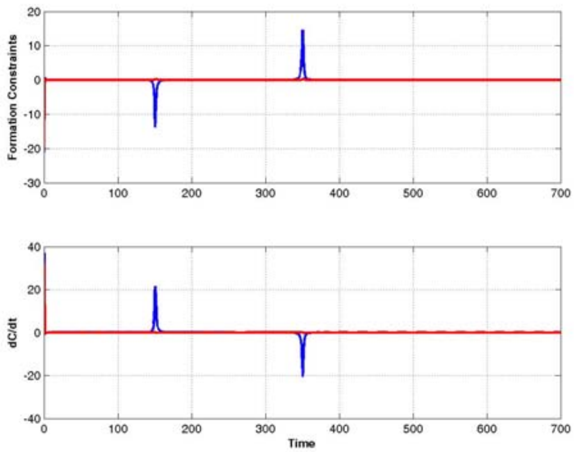

Fig. 6. Constraint function and time derivative.

controller for each member of the formation in a decentralized framework, i.e., the members of the formation are only connected through their constraint functions. In a circle-shaped formation, a vehicle needs only information abour its two neighbors. However, since this applies for every vehicle in the group, the coordination of the entire formation emerges.

Hence, there is no explicit leader or any exogenous system in this design. The dependence on position and velocity measurements requires explicit communication channels between vehicles that coordinate their motion with respect to each other. A decentralized framework for formation control offers some advantages compared to the centralized approach, such as robustness with respect to vehicle loss.

VI. Conclusion

In this paper, we have shown how constraint functions can be designed to hold members of a formation in a desired configuration. The constraints impose forces on the individual vehicles which again maintain the constraints. Further, feedback from the constraints are used to render the system robust again initial position errors during formation assembling, external disturbances and measurement noise. The constraint forces, which can be treated as control laws, are derived in an analytical setting. In particular, we have come up with control laws which maintain the formation structure. Applications have been illustrated by simulations of formation of marine craft.

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