A Survey of Control Allocation Methods for Ships and Underwater Vehicles

Thor I. Fossen
Department of Engineering Cybernetics
Norwegian University of Science and Technology
NO-7491 Trondheim, NORWAY
Email: fossen@ieee.org

Tor A. Johansen
Department of Engineering Cybernetics
Norwegian University of Science and Technology
NO-7491 Trondheim, NORWAY
Email: tor.arne.johansen@itk.ntnu.no

***Abstract*— Control allocation problems for marine vessels can be formulated as optimization problems, where the objective typically is to minimize the use of control effort (or power) subject to actuator rate and position constraints, power constraints as well as other operational constraints. In addition, singularity avoidance for vessels with azimuthing thrusters represent a challenging problem since a non-convex nonlinear program must be solved. This is useful to avoid temporarily loss of controllability in some cases. In this paper, a survey of control allocation methods for overactuated vessels are presented.**

I. Introduction

Overactuated control allocation problems are naturally formulated as optimization problems as one usually wants to take advantage of all available degrees of freedom (DOF) in order to minimize power consumption, drag, tear/wear and other costs related to the use of control, subject to constraints such as actuator position limitations, e.g. [1], [2], [3]. In general, this leads to a constrained optimization problem that is hard to solve using state-of-the-art iterative numerical optimization software at a high sampling rate in a safety-critical real-time system with limiting processing capacity and high demands for software reliability. Still, real-time iterative optimization solutions have been proposed [4], [5], [2], [6], [7]. Explicit solutions can also be found and implemented efficiently by combining simple matrix computations, logic and filtering [8], [9], [10].

Consider a ship or underwater vehicle [11]:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \tag{1}$$

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) = \boldsymbol{\tau} \tag{2}$$

that are controlled by designing a feedback control law of *generalized control forces*:

$$\boldsymbol{\tau} = \mathbf{B}(\boldsymbol{\alpha})\mathbf{u} \in \mathbb{R}^n \tag{3}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^p$ is a vector azimuth angles and $\mathbf{u} \in \mathbb{R}^r$ are *actuator commands*. For ships, some thruster can be rotated an angle about the z-axis and produce force components in the x- and y-directions. This gives additional control inputs $\boldsymbol{\alpha}$ which must be computed by the control allocation algorithm.

The control law uses feedback from position/attitude $\boldsymbol{\eta} = [x, y, z, \phi, \theta, \psi]^\top$ and velocity $\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ as shown in Figure 1.

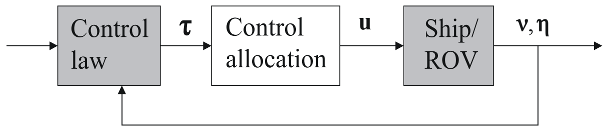

Fig. 1. Block diagram showing the control allocation block in a feedback control system.

For marine vessels in n DOF it is necessary to distribute the generalized control forces $\boldsymbol{\tau}$ to the actuators in terms of control inputs $\boldsymbol{\alpha}$ and $\mathbf{u}$. Consider (3) where $\mathbf{B}(\boldsymbol{\alpha}) \in \mathbb{R}^{n\times r}$ is the input matrix. If $r > n$, and you have control forces both in the x- and y-directions, this is an *overactuated control* problem. Similarly, the case $r < n$ is referred to as an *underactuated control* problem.

Computation of $\boldsymbol{\alpha}$ and $\mathbf{u}$ from $\boldsymbol{\tau}$ is a model-based optimization problem which in its simplest form is unconstrained while physical limitations like input amplitude and rate saturations imply that a constrained optimization problem must be solved. Another complication is actuators that can be rotated at the same time as they produce control forces. An example is azimuth thrusters on an offshore supply vessel. This increases the number of available controls from r to $r + p$.

II. Actuator Models

The control force due to a propeller, a rudder, or a fin can be written (assuming linearity):

$$F = k\ u \tag{4}$$

where k is the force coefficient and u is the control input depending on the actuator considered; see Table I. The linear model $F = ku$ can also be used to describe nonlinear monotonic control forces. For instance, if the rudder force F is quadratic in rudder angle δ, that is $F = k\ \delta\,|\delta|$, the choice $u = \delta\,|\delta|$ which has a unique inverse $\delta = \mathrm{sign}(u)\sqrt{|u|}$ satisfies (4).

For marine vessels the most common actuators are [11]:

- **Main propeller** are mounted aft of the hull usually in conjunction with rudders. They produce the necessary force in the x-direction needed for transit.
- **Tunnel thrusters** are transverse thrusters going through the hull of the vessel. The propeller unit is mounted

inside a transverse tube and it produces a force in the y-direction. Tunnel thrusters are only effective at low speed which limits their use to low-speed maneuvering and DP.

- **Azimuth thrusters** are thruster units that can be rotated an angle α about the z-axis and produce two force components in the horizontal plane are usually referred to as azimuthing thrusters. They are usually mounted under the hull of the vessel and the most sophisticated units are retractable. Azimuth thrusters are attractive in dynamic positioning systems since they can produce forces in different directions leading to an overactuated control problem that can be optimized with respect to power and possible failure situations.
- **Aft rudders** are the primary steering device for conventional vessels. They are located aft of the vessel and the rudder force F_y will be a function of the rudder deflection (the drag force in the x-direction is usually neglected in the control analysis). A rudder force in the y-direction will produce a yaw moment which can be used for steering control.
- **Stabilizing fins** are used for damping of vertical vibrations and roll motions. They produce a force F_z in the z-directions which is a function of the fin deflection. For small angles this relationship is linear. Fin stabilizers can be retractable allowing for selective use in bad weather. The lift forces are small at low speed so the most effective operating condition is in transit.
- **Control surfaces** can be mounted at different locations to produce lift and drag forces. For underwater vehicles these could be fins for diving, rolling, and pitching, rudders for steering, etc.

TABLE I

DEFINITION OF ACTUATORS AND CONTROL VARIABLES WHERE $c\alpha = \cos\alpha$ AND $s\alpha = \sin\alpha$.

Actuator	u	α	$\mathbf{f}^\top$
main propellers (longitudinal)	pitch/rpm	–	$[F, 0, 0]$
tunnel thrusters (transverse)	pitch/rpm	–	$[0, F, 0]$
azimuth (rotatable) thruster	pitch/rpm	angle	$F[c\alpha, s\alpha, 0]$
aft rudders	angle	–	$[0, F, 0]$
stabilizing fins	angle	–	$[0, 0, F]$

Table I implies that the forces and moments in 6 DOF due to the force vector $\mathbf{f} = [F_x, F_y, F_z]^\top$ can be written:

$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \times \mathbf{f} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \\ F_z l_y - F_y l_z \\ F_x l_z - F_z l_x \\ F_y l_x - F_x l_y \end{bmatrix} \tag{5}$$

where $\mathbf{r} = [l_x, l_y, l_z]^\top$ are the moment arms. For azimuth thrusters the control force F will be a function of the rotation angle α; see Figure 2. Consequently, an azimuth thruster in the horizontal plane will have two force components $F_x = F\cos\alpha$ and $F_y = F\sin\alpha$, while the main propeller aft of the ship only produces a longitudinal force $F_x = F$, see Table I.

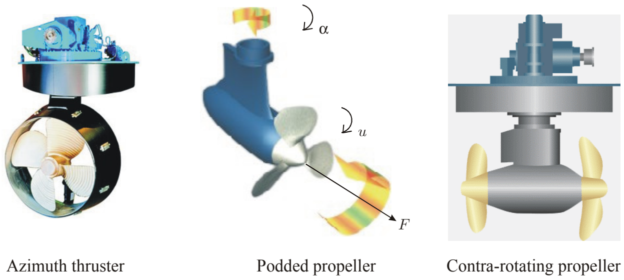

Fig. 2. Propellers that can be rotated an angle α to produce a force F in an arbitrary direction.

A more general representation of control forces and moments is:

$$\boldsymbol{\tau} = \mathbf{T}(\boldsymbol{\alpha})\mathbf{f}, \qquad \mathbf{f} = \mathbf{K}\mathbf{u} \tag{6}$$

where $\mathbf{u} \in \mathbb{R}^r$ and $\boldsymbol{\alpha} \in \mathbb{R}^p$ are control inputs defined as:

$$\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_p]^\top, \qquad \mathbf{u} = [u_1, ..., u_r]^\top \tag{7}$$

and $\mathbf{f} \in \mathbb{R}^r$ is a vector of control forces.

A. Force Coefficient Matrix

The force coefficient matrix $\mathbf{K} \in \mathbb{R}^{r \times r}$ is diagonal:

$$\mathbf{K} = \mathrm{diag}\{k_1, ..., k_r\} \tag{8}$$

B. Actuator Configuration Matrix

The actuator configuration matrix $\mathbf{T}(\boldsymbol{\alpha}) \in \mathbb{R}^{n \times r}$ is defined in terms of a set of column vectors $\mathbf{t}_i \in \mathbb{R}^n$:

$$\mathbf{T}(\boldsymbol{\alpha}) = [\mathbf{t}_1, ..., \mathbf{t}_r] \tag{9}$$

In 4 DOF (*surge, sway, roll,* and *yaw*) the column vectors take the following form:

$$\mathbf{t}_i = \underbrace{\begin{bmatrix} \cos\alpha_i \\ \sin\alpha_i \\ -l_{z_i}\sin\alpha_i \\ l_{x_i}\sin\alpha_i - l_{y_i}\cos\alpha_i \end{bmatrix}}_{\text{azimuth thruster}},$$

$$\mathbf{t}_i = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ l_{y_i} \end{bmatrix}}_{\text{main propeller}}, \quad \mathbf{t}_i = \underbrace{\begin{bmatrix} 0 \\ 1 \\ -l_{z_i} \\ l_{x_i} \end{bmatrix}}_{\substack{\text{tunnel thruster} \\ \text{and aft rudder}}}, \quad \mathbf{t}_i = \underbrace{\begin{bmatrix} 0 \\ 0 \\ l_{y_i} \\ 0 \end{bmatrix}}_{\text{stabilizing fin}}$$

This representation is intended for ships since heave and pitch are neglected. It is straightforward to add these modes for underwater vehicles operating in 6 DOF.

III. LINEAR QUADRATIC UNCONSTRAINED CONTROL ALLOCATION

The simplest allocation problem is the one where all control forces are produced by thrusters in fixed directions alone or in combination with rudders and control surfaces, that is:

$$\boldsymbol{\alpha} = \text{constant}, \qquad \mathbf{T} = \mathbf{T}(\boldsymbol{\alpha}) = \text{constant} \tag{10}$$

Assume that the allocation problem is *unconstrained*–i.e., there are no bounds on the vector elements f_i, α_i, and u_i, and their time derivatives. Saturating control and constrained control allocation are discussed in Sections IV–V.

For marine craft where the configuration matrix $\mathbf{T}$ is square or non-square ($r \geq n$), that is there are equal or more control inputs than controllable DOF, it is possible to find an "optimal" distribution of control forces $\mathbf{f}$, for each DOF by using an explicit method. Consider the unconstrained least-squares (LS) optimization problem (Fossen and Sagatun [12]):

$$\begin{array}{l} \min\limits_{\mathbf{f}} \; \left\{ J = \mathbf{f}^\top \mathbf{W} \mathbf{f} \right\} \\ \text{subject to:} \quad \boldsymbol{\tau} - \mathbf{T}\mathbf{f} = \mathbf{0} \end{array} \tag{11}$$

Here $\mathbf{W}$ is a positive definite matrix, usually diagonal, weighting the control forces. For marine craft which have both control surfaces and propellers, the elements in $\mathbf{W}$ should be selected such that using the control surfaces is much more inexpensive than using the propellers.

A. Explicit Solution for $\boldsymbol{\alpha}$ =const. using Lagrange Multipliers

Define the Lagrangian (Fossen [13]):

$$L(\mathbf{f}, \boldsymbol{\lambda}) = \mathbf{f}^\top \mathbf{W} \mathbf{f} + \boldsymbol{\lambda}^\top (\boldsymbol{\tau} - \mathbf{T}\mathbf{f}) \tag{12}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^r$ is a vector of Lagrange multipliers. Consequently, differentiating the Lagrangian L with respect to $\mathbf{f}$, yields:

$$\frac{\partial L}{\partial \mathbf{f}} = 2\mathbf{W}\mathbf{f} - \mathbf{T}^\top \boldsymbol{\lambda} = \mathbf{0} \Rightarrow \mathbf{f} = \frac{1}{2}\mathbf{W}^{-1}\mathbf{T}^\top \boldsymbol{\lambda} \tag{13}$$

Next, assume that $\mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top$ is non-singular such that:

$$\boldsymbol{\tau} = \mathbf{T}\mathbf{f} = \frac{1}{2}\mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} = 2(\mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top)^{-1}\boldsymbol{\tau} \tag{14}$$

Substituting $\boldsymbol{\lambda} = 2(\mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top)^{-1}\boldsymbol{\tau}$ into (13) yields:

$$\mathbf{f} = \mathbf{T}_w^\dagger \boldsymbol{\tau}, \qquad \mathbf{T}_w^\dagger = \mathbf{W}^{-1}\mathbf{T}^\top (\mathbf{T}\mathbf{W}^{-1}\mathbf{T}^\top)^{-1} \tag{15}$$

where $\mathbf{T}_w^\dagger$ is recognized as the *generalized inverse.* For the case $\mathbf{W} = \mathbf{I}$, that is equally weighted control forces, (15) reduces to the *Moore–Penrose pseudo inverse* $\mathbf{T}^\dagger = \mathbf{T}^\top(\mathbf{T}\mathbf{T}^\top)^{-1}$. Since $\mathbf{f} = \mathbf{T}_w^\dagger \boldsymbol{\tau}$, the control input vector $\mathbf{u}$ can be computed from (6) as:

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{T}_w^\dagger \boldsymbol{\tau} \tag{16}$$

Notice that this solution is valid for all $\boldsymbol{\alpha}$ but not optimal with respect to a time-varying $\boldsymbol{\alpha}$ (only $\mathbf{f}$).

B. Explicit Solution for Varying $\boldsymbol{\alpha}$ using Lagrange Multipliers

In the unconstraint case a time-varying $\boldsymbol{\alpha}$ can be handled by using an *extended thrust representation* similar to Sørdalen [8]. Consider a horizontal plane model (surge, sway, and yaw) of a marine vessel:

$$\boldsymbol{\tau} = \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \\ -l_{y_1} & l_{x_1} & \cdots & -l_{y_r} & l_{x_r} \end{bmatrix} \begin{bmatrix} F_{x_1} \\ F_{y_1} \\ \vdots \\ F_{x_r} \\ F_{y_r} \end{bmatrix} \tag{17}$$

where the thrust $f_i = k_i u_i$ has been decomposed into a surge and sway force component:

$$F_{x_i} = f_i \cos\alpha_i, \qquad F_{y_i} = f_i \sin\alpha_i, \qquad (i = 1...r) \tag{18}$$

This model is in the form $\boldsymbol{\tau} = \mathbf{T}\mathbf{f}_e$ where $\mathbf{T}$ is constant and $\mathbf{f}_e$ denotes the extended thrust vector in (17). Hence, we can compute $\mathbf{f}_e$ by using the generalized inverse, that is $\mathbf{f}_e = \mathbf{T}_w^\dagger \boldsymbol{\tau}$. The optimal azimuth angles and thrust are then found as:

$$\alpha_i = \tan^{-1}\left(\frac{F_{y_i}}{F_{x_i}}\right) \tag{19}$$

$$u_i = \frac{1}{k_i}\sqrt{F_{x_i}^2 + F_{y_i}^2} \tag{20}$$

The main problem is that the optimal solution for α_i can jump at each sample which requires proper filtering. In the next sections, we propose other solutions to this problem.

IV. Linear Quadratic Constrained Control Allocation

In industrial systems it is important to minimize the power consumption by taking advantage of the additional control forces in an overactuated control problem. From a safety critical point of view it is also important to take into account actuator limitations like saturation, tear and wear as well as other constraints such as forbidden sectors, overload of the power system, and the risk for power blackout. In general this leads to a *constrained* optimization problem.

A. Explicit Solution for using Piecewise Linear Functions (Nonrotatable Actuators)

An explicit solution approach for parametric quadratic programming has been developed by Tøndel *et al.* [14] while applications to marine vessels are presented by Johansen *et al.* [15]. In this work the constrained optimization problem is formulated as:

$$\begin{array}{l} \min\limits_{\mathbf{f},\mathbf{s},\bar{f}} \left\{ J = \mathbf{f}^\top \mathbf{W} \mathbf{f} + \mathbf{s}^\top \mathbf{Q} \mathbf{s} + \beta \bar{f} \right\} \\ \text{subject to:} \\ \quad \mathbf{T}\mathbf{f} = \boldsymbol{\tau} + \mathbf{s} \\ \quad \mathbf{f}_{\min} \leq \mathbf{f} \leq \mathbf{f}_{\max} \\ \quad -\bar{f} \leq f_1, f_2, ...f_r \leq \bar{f} \end{array} \tag{21}$$

where $\mathbf{s} \in \mathbb{R}^n$ is a vector of *slack variables.* The first term of the criterion corresponds to the LS criterion (11), while the third term is introduced to minimize the largest force $\bar{f} = \max_i |f_i|$ among the actuators. The constant $\beta \geq 0$ controls the relative weighting of the two criteria. This formulation ensures that the constraints $f_i^{\min} \leq f_i \leq f_i^{\max}$ $(i = 1, ..., r)$ are satisfied, if necessary by allowing the resulting generalized force $\mathbf{T}\mathbf{f}$ to deviate from its specification $\boldsymbol{\tau}$. To achieve accurate generalized force, the slack variable should be close to zero. This is obtained by choosing the weighting matrix $\mathbf{Q} \gg \mathbf{W} > 0$. Moreover, saturation and other constraints are handled in an optimal manner by minimizing the combined criterion (21).

Let $\mathbf{p} = [\boldsymbol{\tau}^\top, \mathbf{f}_{\min}^\top, \mathbf{f}_{\max}^\top, \beta]^\top \in \mathbb{R}^{n+2r+1}$ denote the parameter vector and $\mathbf{z} = [\mathbf{f}^\top, \mathbf{s}^\top, \bar{f}]^\top \in \mathbb{R}^{r+n+1}$. Hence, it is straightforward to see that the optimization problem (21) can be reformulated as a QP problem:

$$\begin{array}{c}\min_{\mathbf{z}} \left\{ J = \mathbf{z}^\top \mathbf{\Phi} \mathbf{z} + \mathbf{z}^\top \mathbf{R} \mathbf{p} \right\} \\ \text{subject to:} \\ \mathbf{A}_1 \mathbf{z} = \mathbf{C}_1 \mathbf{p} \\ \mathbf{A}_2 \mathbf{z} \leq \mathbf{C}_2 \mathbf{p} \end{array} \tag{22}$$

where:

$$\begin{aligned}
\mathbf{\Phi} &= \begin{bmatrix} \mathbf{W} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times 1} \\ \mathbf{0}_{n\times r} & \mathbf{Q} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times r} & \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \\
\mathbf{R} &= \begin{bmatrix} \mathbf{0}_{(r+n+1)\times(n+2r)} & \begin{bmatrix} \mathbf{0}_{(r+n)\times 1} \\ 1 \end{bmatrix} \end{bmatrix} \\
\mathbf{A}_1 &= \begin{bmatrix} \mathbf{T} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} \end{bmatrix} \\
\mathbf{C}_1 &= \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0}_{n\times(2r+1)} \end{bmatrix} \\
\mathbf{C}_2 &= \begin{bmatrix} \mathbf{0}_{r\times n} & -\mathbf{I}_{r\times r} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times 1} \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} & \mathbf{I}_{r\times r} & \mathbf{0}_{r\times 1} \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times 1} \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times 1} \end{bmatrix} \\
\mathbf{A}_2 &= \begin{bmatrix} -\mathbf{I}_{r\times r} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times 1} \\ \mathbf{I}_{r\times r} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times 1} \\ \mathbf{I}_{r\times r} & \mathbf{0}_{r\times n} & \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ \mathbf{I}_{r\times r} & \mathbf{0}_{r\times n} & -\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix}
\end{aligned}$$

Since $\mathbf{W} > 0$ and $\mathbf{Q} > 0$ this is a convex quadratic program in $\mathbf{z}$ parametrized by $\mathbf{p}$. Convexity guarantees that a global solution can be found. The optimal solution $\mathbf{z}^*(\mathbf{p})$ is a continuous piecewise linear function $\mathbf{z}^*(\mathbf{p})$ defined on any subset:

$$\mathbf{p}_{\min} \leq \mathbf{p} \leq \mathbf{p}_{\max} \tag{23}$$

of the parameter space. Moreover, an exact representation of this piecewise linear function can be computed off-line using multi-parametric QP algorithms [14]. Consequently, it is not necessary to solve the QP (21) in real time for the current value of $\boldsymbol{\tau}$, and the parameters $f_{\min}$, $f_{\max}$, and β, if they are allowed to vary. In fact it suffices to evaluate the known piecewise linear function $\mathbf{z}^*(\mathbf{p})$ as a function of the given parameter vector $\mathbf{p}$ which can be done efficient with a small amount of computations. For details on the implementation aspects of the mp-QP algorithm; see Johansen *et al.* [14] and references therein. An on-line control allocation algorithm is presented in Tøndel *et al.* [16]

B. Explicit Solution for using Piecewise Linear Functions (Azimuthing Thrusters and Rudders)

An extension of the mp-QP algorithm to marine vessels equipped with azimuthing thrusters and rudders has been done by Johansen *et al.* [17]. A propeller with a rudder can produce a thrust vector within a range of directions and magnitudes in the horizontal plane for low-speed maneuvering and dynamic positioning. The set of attainable thrust vectors is non-convex because significant lift can be produced by the rudder only with forward thrust. The attainable thrust region can, however, be decomposed into a finite union of convex polyhedral sets. A similar decomposition can be made for azimuthing thrusters including forbidden sectors. Hence, this can be formulated as a mixed-integer-like convex quadratic programming problem and by using arbitrarily number of rudders as well as thrusters and other propulsion devices can be handled. Actuator rate and position constraints are also taken into account. Using a multi-parametric quadratic programming software, an explicit piecewise linear representation of the least-squares optimal control allocation law can be pre-computed. The method is illustrated using a scale model of a supply vessel in test basin, see Johansen *et al.* [17] for details.

C. Explicit Solutions based on Minimum Norm and Null-Space Methods (Nonrotatable Actuators)

In flight and aerospace control systems, the problems of control allocation and saturating control have been addressed by Durham [3],[18], and [19]. They also propose an explicit solution to avoid saturation referred to as the "direct method". By noticing that there are infinite combinations of admissible controls that generates control forces on the boundary of the closed subset of attainable controls, the "direct method" calculates admissible controls in the interior of the attainable forces as scaled down versions of the unique solutions for force demands. Unfortunately it is not possible to minimize the norm of the control forces on the boundary or some other constraint since the solutions on the boundary are unique. The computational complexity of the algorithm is proportional to the square of the number of controls, which can be problematic in real-time applications.

In Bordignon and Durham [20] the null space interaction method is used to minimize the norm of the control vector when possible, and still access the attainable forces to overcome the drawbacks of the "direct method". This method is also explicit but much more computational intensive. For instance 20 independent controls imply that up to 3.4 billon points have to be checked at each sample. In Durham [21] a computationally simple and efficient method to obtain near-optimal solutions is described. The method is based on prior knowledge of the controls' effectiveness and limits such that precalculation of several generalized inverses can be done.

D. Iterative Solutions

An alternative to the explicit solution could be to use an iterative solution to solve the QP problem (Sørdalen [8]). The drawback with the iterative solution is that several iterations may have to be performed at each sample in order to find the optimal solution. The iterative approach is more flexibility for on-line reconfiguration, as for example a change in $\mathbf{W}$ may require that the explicit solutions are recalculated. Computational complexity is also greatly reduced by a "warm start"– i.e., the numerical solver is initialized with the solution of the optimization problem computed at the previous sample.

Finally, the offline computed complexity and memory requirements may be prohibited for the explicit solution to be applicable to large scale control allocation problems.

V. Nonlinear Constrained Control Allocation (Azimuthing Thrusters)

The control allocation problem for vessels equipped with azimuth thrusters is in general a *non-convex* optimization problem that is hard to solve. The primary constraint is:

$$\boldsymbol{\tau} = \mathbf{T}(\boldsymbol{\alpha})\mathbf{f} \tag{24}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^p$ denotes the azimuth angles. The azimuth angles must be computed at each sample together with the control inputs $\mathbf{u} \in \mathbb{R}^r$ which are subject to both amplitude and rate saturations. In addition, azimuthing thrusters may only operate in feasible sectors $\alpha_{i,\min} \leq \alpha_i \leq \alpha_{i,\max}$ at a limiting turning rate $\dot{\boldsymbol{\alpha}}$. Another problem is that the inverse:

$$\mathbf{T}_w^{\dagger}(\boldsymbol{\alpha}) = \mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha})[\mathbf{T}(\boldsymbol{\alpha})\mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha})]^{-1} \tag{25}$$

may not exist for certain $\boldsymbol{\alpha}$-values due to singularity. The consequence of such a singularity is that no force is produced in certain directions. This may greatly reduce dynamic performance and maneuverability as the azimuth angles can be changed slowly only.

This suggests that the following criterion should be minimized (Johansen *et al.* [7]):

$$\min_{\mathbf{f},\boldsymbol{\alpha},\mathbf{s}} \left\{ J = \sum_{i=1}^{r} \bar{P}_i \left| f_i \right|^{3/2} + \mathbf{s}^{\top}\mathbf{Q}\mathbf{s} + (\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)^{\top}\boldsymbol{\Omega}(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0) + \frac{\varrho}{\varepsilon + \det(\mathbf{T}(\boldsymbol{\alpha})\mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha}))} \right\} \tag{26}$$

subject to

$$\begin{aligned}
&\mathbf{T}(\boldsymbol{\alpha})\mathbf{f} = \boldsymbol{\tau} + \mathbf{s}\\
&\mathbf{f}_{\min} \leq \mathbf{f} \leq \mathbf{f}_{\max}\\
&\boldsymbol{\alpha}_{\min} \leq \boldsymbol{\alpha} \leq \boldsymbol{\alpha}_{\max}\\
&\Delta\boldsymbol{\alpha}_{\min} \leq \boldsymbol{\alpha} - \boldsymbol{\alpha}_0 \leq \boldsymbol{\Delta\alpha}_{\max}
\end{aligned}$$

where:

- $\sum_{i=1}^{r} \bar{P}_i \left| f_i \right|^{3/2}$ represents power consumption where $\bar{P}_i > 0$ $(i = 1, ..., r)$ are positive weights.
- $\mathbf{s}^{\top}\mathbf{Q}\mathbf{s}$ penalizes the error $\mathbf{s}$ between the commanded and achieved generalized force. This is necessary in order to guarantee that the optimization problem has a feasible solution for any $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}_0$. The weight $\mathbf{Q} > 0$ is chosen so large that the optimal solution is $\mathbf{s} \approx \mathbf{0}$ whenever possible.
- $\mathbf{f}_{\min} \leq \mathbf{f} \leq \mathbf{f}_{\max}$ is used to limit the use of force (saturation handling).
- $\boldsymbol{\alpha}_{\min} \leq \boldsymbol{\alpha} \leq \boldsymbol{\alpha}_{\max}$ denotes the feasible sectors of the azimuth angles.
- $\Delta\boldsymbol{\alpha}_{\min} \leq \boldsymbol{\alpha} - \boldsymbol{\alpha}_0 \leq \boldsymbol{\Delta\alpha}_{\max}$ ensures that the azimuth angles do not move to much within one sample taking $\boldsymbol{\alpha}_0$ equal to the angles at the previous sample. This is equivalent to limiting $|\dot{\boldsymbol{\alpha}}|$,–i.e. the turning rate of the thrusters.
- The term:
$$\frac{\varrho}{\varepsilon + \det(\mathbf{T}(\boldsymbol{\alpha})\mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha}))}$$
is introduced to avoid singular configurations given by $\det(\mathbf{T}(\boldsymbol{\alpha})\mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha})) = 0$. To avoid division by zero, $\varepsilon > 0$ is chosen as a small number, while $\varrho > 0$ is scalar weight. A large ϱ ensures high maneuverability at the cost of higher power consumption and vice versa.

The optimization problem (26) is a non-convex nonlinear program and it requires a significant amount of computations at each sample [22]. Consequently, the following two implementation strategies are attractive alternatives to nonlinear program efforts.

A. Dynamic Solution using Lyapunov Methods

In Johansen [23] a control-Lyapunov approach has been used to develop an optimal dynamic control allocation algorithm. The proposed algorithm leads to asymptotic optimality. Consequently, the computational complexity compared to a direct nonlinear programming approach is considerably reduced. This is done by constructing the optimizing control allocation algorithm as a dynamic update law which can be used together with a feedback control system. It is shown that the asymptotically optimal control allocation algorithm in interaction with an exponentially stable trajectory-tracking controller guarantees uniform boundedness and uniform global exponential convergence. A case study addressing low-speed maneuvering of an overactuated ship is used to demonstrate the performance of the control allocation algorithm.

B. Iterative Solutions using Quadratic Programming

The problem (26) can be locally approximated with a *convex* QP problem by assuming that:

1) the power consumption can be approximated by a quadratic term in $\mathbf{f}$, near the last force $\mathbf{f}_0$ such that $\mathbf{f} = \mathbf{f}_0 + \Delta\mathbf{f}$.
2) the singularity avoidance penalty can be approximated by a linear term linearized about the last azimuth angle $\boldsymbol{\alpha}_0$ such that $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha}$.

The resulting QP criterion is (Johansen *et al.* [7]):

$$\min_{\Delta\mathbf{f},\boldsymbol{\Delta\alpha},\mathbf{s}} \left\{ J = (\mathbf{f}_0 + \Delta\mathbf{f})^{\top}\mathbf{P}(\mathbf{f}_0 + \Delta\mathbf{f}) + \mathbf{s}^{\top}\mathbf{Q}\mathbf{s} + \Delta\boldsymbol{\alpha}^{\top}\boldsymbol{\Omega}\Delta\boldsymbol{\alpha} + \frac{\partial}{\partial\boldsymbol{\alpha}}\left(\frac{\varrho}{\varepsilon + \det(\mathbf{T}(\boldsymbol{\alpha})\mathbf{W}^{-1}\mathbf{T}^{\top}(\boldsymbol{\alpha}))} \right)\Bigg|_{\boldsymbol{\alpha}_0} \Delta\boldsymbol{\alpha} \right\} \tag{27}$$

subject to

$$\begin{aligned}
&\mathbf{s} + \mathbf{T}(\boldsymbol{\alpha}_0)\Delta\mathbf{f} + \frac{\partial}{\partial\boldsymbol{\alpha}}\left(\mathbf{T}(\boldsymbol{\alpha}_0)\mathbf{f}\right)\big|_{\boldsymbol{\alpha}_0,\mathbf{f}_0} \boldsymbol{\Delta\alpha} = \boldsymbol{\tau} - \mathbf{T}(\boldsymbol{\alpha}_0)\mathbf{f}_0\\
&\mathbf{f}_{\min} - \mathbf{f}_0 \leq \mathbf{f} \leq \mathbf{f}_{\max} - \mathbf{f}_0\\
&\boldsymbol{\alpha}_{\min} - \boldsymbol{\alpha}_0 \leq \boldsymbol{\Delta\alpha} \leq \boldsymbol{\alpha}_{\max} - \boldsymbol{\alpha}_0\\
&\Delta\boldsymbol{\alpha}_{\min} \leq \boldsymbol{\Delta\alpha} \leq \boldsymbol{\Delta\alpha}_{\max}
\end{aligned}$$

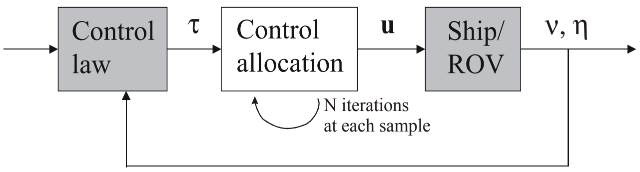

Fig. 3. Control allocation using an iterative solution.

The convex QP problem (27) can be solved by using standard software for numerical optimization.

C. Iterative Solutions using Linear Programming

Linear approximations to the thrust allocation problem has been discussed by [5] and Lindfors [4]. In [4] the azimuth thrust constraints:

$$|f_i| = \sqrt{(f_i \cos\alpha_i)^2 + (f_i \sin\alpha_i)^2} \le f_i^{\max} \qquad (28)$$

are represented as circles in the $(f_i \cos\alpha_i, f_i \sin\alpha_i)$-plane. The nonlinear program is transformed to a linear programming (LP) problem by approximating the azimuth thrust constraints by straight lines forming a polygon. If 8 lines are used to approximate the circles (octagons), the worst case errors will be less than ± 4.0 %. The criterion to be minimized is a linear combination of $|\mathbf{f}|$, that is magnitude of force in the x- and y-directions, weighted against the magnitudes $|\sqrt{(f_i \cos\alpha_i)^2 + (f_i \sin\alpha_i)^2}|$ representing azimuth thrust. Hence, singularities and azimuth rate limitations are not weighted in the cost function. If these are important, the QP formulation should be used.

D. Explicit Solution using the Singular Value Decomposition and Filtering Techniques

An alternative method to solve the constrained control allocation problem is to use the singular value decomposition (SVD) and a filtering scheme to control the azimuth directions such that they are aligned with the direction where most force is required, paying attention to singularities (Sørdalen *et al.* [8]). Results from sea trials have been presented in Sørdalen [8]. A similar technique using the damped-least squares algorithm has been reported in Berge and Fossen [9] where the results are documented by controlling a scale model of a supply vessel equipped with four azimuth thrusters.

VI. Conclusions

A survey of methods for control allocation of overactuated marine vessels have been presented. Both implicit and explicit methods formulated as optimization problems have been discussed. The objective has been to minimize the use of control effort (or power) subject to actuator rate and position constraints, power constraints as well as other operational constraints.

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