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Motion Control of Marine Craft Using Virtual Positional and Velocity Constraints

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***Abstract*—Recently, [1] put forward a multi-body representation for modeling and control of formations of marine craft. In this paper, the same line of reasoning is taken one step further and the method is generalized for the cases that include speed assignment tasks. The proposed algorithm is then applied to design a novel path-following controller, in which the control objectives are formulated using virtual positional and velocity constraints, and the control problem is regarded as a constrained multi-body system. Global exponential stability of the error system is proved using the Lyapunov method. Also, global boundedness of the unactuated dynamics is demonstrated in the case of underactuation.**

I. INTRODUCTION

Nowadays, marine craft play a central role in the modern world; in parallel with this, motion control has attracted a great deal of attention to design controllers to cope with nonlinearities and external disturbances. Marine craft have been controlled using various linear and nonlinear control methodologies; e.g. see [2]. Conducting an inspection, one realizes that a nonlinear model of the system, achieved using the Lagrangian formulation, is usually chosen and standard control methodologies are applied to find stabilizing control laws. This framework is mathematical and it does not take advantage of physical information, subtly hidden behind the equations of motion. Linearization about one operating point also destroys geometric properties while physical characteristics can be exploited to improve the performance.

One circle of thinking for motion control of mechanical systems is to conceive them constrained to some manifolds/bodies using virtual springs and dampers. Then, forces coming out of imaginary springs and dampers would keep the system moving as it should. The concept of virtual constraints was first introduced in [3] in the field of *force* control of robot manipulators interacting with the environment.

In [1], a method to form a desired configuration by a group of surface vessels is proposed. The main insight is that control objectives are reformulated in terms of virtual mechanical constraints; then the force of constraints is obtained using the Lagrange multiplier method, which establishes and maintains the desired formation. The paper considers *fully* actuated craft. Also, the interconnection between bodies is expressed in terms of *position* variables.

The aim of this paper is to generalize the results of the early paper [1] for situations involving speed assignment tasks. The generalization is made in a way that both positional and velocity restrictions are covered simultaneously. The method is general and systematic, and it provides a closed-form solution. Moreover, control laws by this method can be physically interpretable. To demonstrate the effectiveness of the proposed method, a nonlinear controller is designed for marine surface craft asked to follow a parameterized path, where the geometric and dynamic tasks of the maneuvering problem are achieved globally and exponentially. In the case of underactuation, a dynamic controller is derived, proving global boundedness of the unactuated dynamics.

The outline of the paper is as follows. Section II is a preliminary review of Lagrangian mechanics and constrained multi-body systems. The control method is proposed in Section III where virtual mechanical constraints are stabilized using two methods. Then, the algorithm is specified for path following. A simulation study is presented in Section V.

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II. REVIEW OF LAGRANGIAN MECHANICS

In this paper, we briefly explain the fundamentals of the Lagrangian formulation and the way it handles constrained systems, on which the proposed control method is based.

A. Lagrangian Equations of Motion

Let $\boldsymbol{q} \in \mathbb{R}^n$ be the vector of generalized coordinates, uniquely describing an n degrees-of-freedom (n-DOF) mechanical system in an n-dimensional configuration space, that is a smooth manifold and locally diffeomorphic; then $\dot{\boldsymbol{q}}$ is the vector of generalized velocities. For a system with inertia matrix $\mathcal{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$, kinetic energy $\mathcal{T}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\mathcal{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ and potential energy $\mathcal{V}(\boldsymbol{q})$, the Lagrangian is

$$\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{V}(\boldsymbol{q}) \tag{1}$$

Hamilton's principle relates the Lagrangian to the motion of the unforced system whereas the Lagrange-d'Alembert principle generalizes Hamilton's principle in the presence of external forces, $\boldsymbol{\tau}$. The Euler-Lagrange equations are expressed as:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_i} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_i} = \tau_i \qquad i = 1,\cdots,n \tag{2}$$

which leads to

$$\mathcal{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau} \tag{3}$$

where the vector field $\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is easily obtained by inspection. This is the typical structure for models of robot manipulators, mobile robots [4], and marine craft [2]. However, the system may be restricted to a manifold by some constraints. The Lagrangian framework deals with constraints in a way that is briefly explained in the next subsections.

B. Holonomic Constraints

We assume that the system is subject to k *geometric* constraints in the vector form:

$$\boldsymbol{G}(\boldsymbol{q}, t) = 0 \tag{4}$$

In mechanics, geometric constraints (4), which depend only on generalized coordinates, is called holonomic constraints. Holonomic constraints arise out of interconnections of bodies in mechanical systems. It has been shown in [5], that Hamilton's generalized principle handles such constraints by modifying the Lagrangian to

$$\mathcal{L}^*(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\lambda}_G^T \boldsymbol{G}(\boldsymbol{q}, t), \qquad \boldsymbol{\lambda}_G \in \mathbb{R}^k \tag{5}$$

Then applying the Euler-Lagrange equations to the augmented Lagrangian $\mathcal{L}^*$ over the extended set of free coordinates $\{\boldsymbol{q}, \boldsymbol{\lambda}_G\}$ gives the equations of motion. This is the well-known Lagrange multiplier rule, which leads to (3) with

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{ext} + \boldsymbol{\tau}_c \tag{6}$$

where $\boldsymbol{\tau}_{ext}$ denotes the external forces while the constraint forces are computed using

$$\boldsymbol{\tau}_c = -\mathcal{W}_G^T(\boldsymbol{q}, t)\boldsymbol{\lambda}_G \tag{7}$$

in which $\mathcal{W}_G = \partial \boldsymbol{G}/\partial \boldsymbol{q} \in \mathbb{R}^{k\times n}$. Holonomic constraints confine the system geometrically; that is, those points where the constraints are violated are not accessible for the system. Hence, holonomic constraints affect controllability. Moreover, geometric confinements (4) state that there exists a reduced-order configuration space in which all coordinates are independent and all points are accessible. Therefore, one may solve (4) to find an explicit solution for dependent coordinates and eliminate coordinate dependency. However, direct elimination is not recommended in that it may cause algebraic singularities, multiple solutions, or local validity [6].

C. Nonholonomic Constraints

In addition to geometric constraints, a system may involve m first-order *kinematic* constraints, represented in the vector form

$$\boldsymbol{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0 \tag{8}$$

If (8) is integrable, it can be expressed in terms of the generalized coordinates and time; it is thus intrinsically holonomic. Otherwise, (8) depends on velocity variables, and it is said to be nonholonomic. In this paper, we draw attention to a class of kinematic constraints in which the velocity vector appears linearly

$$\boldsymbol{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \triangleq \mathcal{A}(\boldsymbol{q}, t)\dot{\boldsymbol{q}} + \boldsymbol{B}(\boldsymbol{q}, t) \tag{9}$$

where $\mathcal{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^m$. The constraint (9) is called *Pfaffian* if it is driftless and time-independent; i.e. $\mathcal{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$. Pfaffian constraints are very common in vehicles without skidding. In spacecraft, nonholonomy may stem from conservation of angular momentum. As stated in [7], underactuated mechanical systems may also introduce nonholonomic constraints but at acceleration level.

Considering Pfaffian constraints, one can find the matrix $\mathcal{H}(\boldsymbol{q}) \in \mathbb{R}^{n\times(n-m)}$ that describes the null-space of $\mathcal{A}(\boldsymbol{q})$; i.e., $\mathcal{A}(\boldsymbol{q})\mathcal{H}(\boldsymbol{q}) = 0$. Thus, the admissible $\dot{\boldsymbol{q}}$ is expressed in terms of the columns of $\mathcal{H}(\boldsymbol{q})$; that is

$$\dot{\boldsymbol{q}} = \mathcal{H}(\boldsymbol{q})\boldsymbol{w}, \qquad \boldsymbol{w} \in \mathbb{R}^{n-m} \tag{10}$$

Equation (10), which describes a nonlinear and affine system, is called the kinematic model of a nonholonomic system and $\boldsymbol{w}$ may be interpreted as the virtual input of the system. Notice that the system is underactuted from a control viewpoint, implying a close relation between nonholonomy and underactuation. The dependency of $\dot{\boldsymbol{q}}$ is removed if (10) is differentiated and substituted in (3), which results in

$$\mathcal{H}^T \mathcal{M}\mathcal{H}\dot{\boldsymbol{w}} + \mathcal{H}^T(\boldsymbol{n} + \mathcal{M}\dot{\mathcal{H}}\boldsymbol{w}) = \mathcal{H}^T\boldsymbol{\tau} \tag{11}$$

Equation (11) is usually called the dynamic model of a nonholonomic system [8], leading to a canonical state-space form in conjunction with (10) . Note that although for a naturally constrained system the dynamic model is always attainable, it may be unfeasible for virtually constrained systems.

Compared to holonomic constraints, nonholonomy imposes no geometric restriction on the configuration space, but they prevent the system from moving in directions which constraints are violated. It implies that nonholonomy does not affect controllability of the system. It thus suggests studying accessibility conditions [9] of the kinematic model to judge whether constraints (8) are holonomic. Accordingly, the degree of rank deficiency of the accessibility distribution shows the number of integrable (holonomic) constraints.

Dealing with nonholonomic systems is harder than the holonomic case. This is due to the confusion over the generalization of Hamilton's principle and the use of the Lagrange multipliers for the general nonholonomic constraints (8). The confusion misled [10] and [11] into generalizing Hamilton's principle for (8) akin to (5), but this generalization resulted in a wrong outcome for the Pfaffian constraints, as explained in [12] and [4]. It is while [13] encounters the same problem and believes the Lagrange multiplier method is applicable for the general case (5). Still, [14] and [15] acknowledge the fallacy of the Lagrange multiplier method for the general case and propose a multiplier-free approach to tackle nonholonomy.

Fortunately, in [16], the author neatly demystifies the enigma and proposes the most convincing answer to the bewildering problem of nonholonomic constraints. To put it simply, general nonholonomic constraints (5) are completely outside of the scope of any principle based on d'Alembert's principle, and any generalization according to it is baseless. It is valuable to note that d'Alembert's principle of virtual work consist in the fact that constraint forces do no work under any virtual displacements; it is the fundamental principle in analytical mechanics that Hamilton's principle can be extracted from. Reference [16] also states that d'Alembert's generalized principle can be applied to holonomic constraints, and undoubtedly the Lagrange multiplier technique is workable, which is not an extraordinary result, though. The Lagrange multiplier method is surprisingly no longer valid for the nonholonomic linear-in-velocity constraints (9) while they can be treated by d'Alembert's *basic* principle. Consequently, for a mechanical system expressed by Lagrangian mechanics and subject to kinematic constraints (9), (6) is still valid with

$$\boldsymbol{\tau}_c = -\mathcal{A}^T(\boldsymbol{q},t)\boldsymbol{\lambda}_K, \qquad \boldsymbol{\lambda}_K \in \mathbb{R}^m \tag{12}$$

The constraint force arising from the linear-in-velocity constraints (9) takes the same form as the holonomic constraints do even though the way (12) is derived from is different.

It is worthwhile mentioning that constraints (4) and (9) satisfy the principle of virtual work whereas the general case (8) may not. Those constraints whose respective forces produce work under virtual displacements are said to be *nonideal*. It has been shown [17], [18] that in case the constraints are nonideal and its first derivative is linear in acceleration (i.e. $\mathcal{B}(\boldsymbol{q},\dot{\boldsymbol{q}},t)\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = 0$), the constraint force (12) will change to

$$\boldsymbol{\tau}_c = -\mathcal{A}^T(\boldsymbol{q},t)\boldsymbol{\lambda}_K + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}},t), \qquad \boldsymbol{C} \in \mathbb{R}^n \tag{13}$$

where the vector $\boldsymbol{C}$ specifies the work done by nonideal constraints and has to be chosen based on the analyst's knowledge for the specific system. Notice that in the present paper, we pay attention to ideal constraints.

D. Holonomically & Nonholonomically Constrained Systems

If a system is subject to m nonholonomic linear-in-velocity constraints $\boldsymbol{K}$ and k holonomic constraints $\boldsymbol{G}$, the force vector that holds both constraints is given by

$$\boldsymbol{\tau}_c = -\mathcal{W}^T\boldsymbol{\lambda} \tag{14}$$

in which $\mathcal{W} = [\mathcal{A}^T, \mathcal{W}_G^T]^T \in \mathbb{R}^{n\times(m+k)}$ is the Jacobian matrix and $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_K^T, \boldsymbol{\lambda}_G^T]^T \in \mathbb{R}^{m+k}$ is the multiplier vector.

III. Overview Of The Control Methodology

In this section, the general perspective of the proposed method is given. Fig. 1 displays a system subject to both positional (holonomic) and velocity (nonholonomic) restrictions. These constraints may be set on the system either by nature (called natural constraints), or by virtue of control problems (called virtual constraints). The system along with the constraints forms the "constrained system". The constraint functions are then unified and the forces that hold the constraints are derived using the model. In the "constraint stabilization" block, unification is done at the kinematic level whereas force derivation is involved with kinetics. The constraint manifold is made invariant so that all trajectories from different initial conditions asymptotically converge to it. In what follows, first some examples of virtual constraints in maritime applications are given. Then, two methods for constraint stabilization are provided.

A. Virtual Mechanical Constraints

Control objectives in different strategies can be viewed as virtual mechanical constraints. Trajectory tracking is the problem of forcing the states of the system to asymptotically track desired time-varying signals, $\boldsymbol{q}_d(t)$. This scenario leads

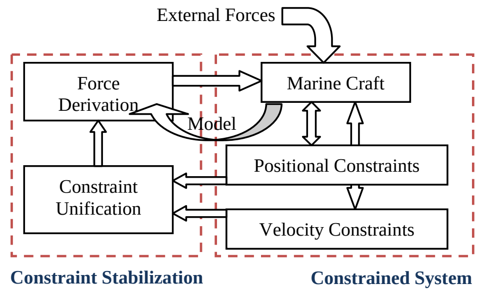

Fig. 1: Diagram of the proposed framework.

to a set of holonomic constraints: $\boldsymbol{H}_{tt}(\boldsymbol{q},t) = \boldsymbol{q}(t) - \boldsymbol{q}_d(t)$. Alternatively, path following (PF) is the problem of forcing a craft to asymptotically converge to and follow a given path. The PF problem can be decomposed into two tasks [19]; one is the geometric task that is to reduce the distance of the craft to the path, usually expressed as a constraint on the course angle, ψ, for surface marine craft. The other is the dynamic task defined on the speed of the system, u. Thus, 2-D PF imposes one holonomic constraint and one nonholonomic constraint as: $\boldsymbol{H}_{pf}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = [u(t) - u_d(t), \psi - \psi_d(t)]^T$ in which $u_d(t)$ and $\psi_d(t)$ are the desired signals.

Formation control is the problem of forcing a group of vehicles to form a desired geometric configuration. This can be formulated by relative distance among members, bringing in holonomic constraints: (e.g. for two craft) $\boldsymbol{H}_{f1}(\boldsymbol{q}) = \|\boldsymbol{q}_1 - \boldsymbol{q}_2\|$ or alternatively: $\boldsymbol{H}_{f2}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = [u_1 - u_2, \psi_1 - \psi_2, e_1 - e_2]^T$ which imposes a combination of holonomic and nonholonomic constraints. e_i denotes the lateral distance between craft.

B. Constraint Stabilization

Inspired by techniques for forward dynamics simulation of constrained multi-body systems, stabilizing controllers for motion control of marine craft are derived. The objective of constraint stabilization is to make the constraint manifold invariant such that all trajectories from all initial conditions converge to it. Two methods are presented. In the first method, both constraints are taken into the acceleration level and then they merge together to find the constraint force vector. Alternatively, unification can take place at the velocity level when holonomic constraints are first stabilized; then, the unified constraint function is utilized to derive the constraint forces. Unification methods are now clarified.

Method I: We follow the same lines of thinking as [1], and make $\boldsymbol{G}(\boldsymbol{q},t)$ an attractive manifold by taking $\boldsymbol{x}_1 = \boldsymbol{G}$, $\boldsymbol{x}_2 = \dot{\boldsymbol{G}}$, $\boldsymbol{x} \triangleq [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T$, and

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0_{k\times k} & \mathcal{I}_k \\ 0_{k\times k} & 0_{k\times k} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0_k \\ \boldsymbol{u}_{1G} \end{bmatrix} \tag{15}$$

Then, $\boldsymbol{u}_{1G}$ is chosen in a way such that (15) is globally asymptotically/exponentially stable (GAS/GES) at the origin. One choice may be $\boldsymbol{u}_{1G} = -\mathcal{L}_{1G}\boldsymbol{x}_1 - \mathcal{L}_{2G}\boldsymbol{x}_2$ which leads to

$$\ddot{\boldsymbol{G}} + \mathcal{L}_{2G}\dot{\boldsymbol{G}} + \mathcal{L}_{1G}\boldsymbol{G} = 0 \tag{16}$$

where $\mathcal{L}_{2G} = \mathcal{L}_{2G}^T > 0, \mathcal{L}_{1G} = \mathcal{L}_{1G}^T > 0$ are $k \times k$ real matrices. Indeed, fulfillment of $\boldsymbol{G} = 0, \forall t$ requires $\dot{\boldsymbol{G}} = 0, \forall t$. Equation (16) is equivalent to

$$\mathcal{W}_G \ddot{\boldsymbol{q}} + (\dot{\mathcal{W}}_G + \mathcal{L}_{2G}\mathcal{W}_G)\dot{\boldsymbol{q}} + \mathcal{L}_{1G}\boldsymbol{G} + \boldsymbol{G}^{tt} + \mathcal{L}_{2G}\boldsymbol{G}^t = 0 \quad (17)$$

in which $\boldsymbol{G}^t = \partial \boldsymbol{G}/\partial t \in \mathbb{R}^k$ and $\boldsymbol{G}^{tt} = \partial^2 \boldsymbol{G}/\partial t^2 \in \mathbb{R}^k$. Similarly, the nonholonomic constraint manifold, see (9), is made attractive using $\dot{\boldsymbol{K}} = \boldsymbol{u}_{1K}$. Picking $\boldsymbol{u}_{1K} = -\mathcal{L}_K \boldsymbol{K}$, where $\mathcal{L}_K = \mathcal{L}_K^T > 0$ and $\mathcal{L}_K \in \mathbb{R}^{m\times m}$, results in

$$\mathcal{A}\ddot{\boldsymbol{q}} + (\dot{\mathcal{A}} + \mathcal{L}_k \mathcal{A})\dot{\boldsymbol{q}} + \dot{\boldsymbol{B}} + \mathcal{L}_K \boldsymbol{B} = 0 \quad (18)$$

Lumping (17) and (18) together, a unified equation is obtained

$$\mathcal{W}\ddot{\boldsymbol{q}} + \begin{bmatrix} \dot{\mathcal{A}} + \mathcal{L}_k \mathcal{A} \\ \dot{\mathcal{W}}_G + \mathcal{L}_{2G}\mathcal{W}_G \end{bmatrix} \dot{\boldsymbol{q}} + \begin{bmatrix} \dot{\boldsymbol{B}} + \mathcal{L}_K \boldsymbol{B} \\ \mathcal{L}_{1G}\boldsymbol{G} + \boldsymbol{G}^{tt} + \mathcal{L}_{2G}\boldsymbol{G}^t \end{bmatrix} = 0 \quad (19)$$

If the acceleration $\ddot{\boldsymbol{q}}$ is computed using the model of the system, an equation to find the constraint forces are derived.

Method II: The second method, inspired by [6], is similar to the first method, but the unification happens at the velocity level. To this end, the holonomic constraints are first stabilized as a first-order system; i.e., $\dot{\boldsymbol{G}} = \boldsymbol{u}_{2G}$. Choosing $\boldsymbol{u}_{2G} = -\mathcal{P}_G \boldsymbol{G}$, it follows that:

$$\dot{\boldsymbol{G}} + \mathcal{P}_G \boldsymbol{G} = 0 \Rightarrow \mathcal{W}_G \dot{\boldsymbol{q}} + \boldsymbol{G}^t + \mathcal{P}_G \boldsymbol{G} = 0 \quad (20)$$

in which $\mathcal{P}_G = \mathcal{P}_G^T > 0$ is a $k \times k$ real matrix. Putting (20) and (9) into a unified form, one gets

$$\boldsymbol{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \triangleq \mathcal{W}(\boldsymbol{q}, t)\dot{\boldsymbol{q}} + \boldsymbol{a}(\boldsymbol{q}, t) \quad (21)$$

$$\boldsymbol{a}(\boldsymbol{q}, t) = \begin{bmatrix} \boldsymbol{\mathcal{B}}(\boldsymbol{q}, t) \\ \boldsymbol{G}^t(\boldsymbol{q}, t) + \mathcal{P}_G \boldsymbol{G}(\boldsymbol{q}, t) \end{bmatrix} \quad (22)$$

In (21), $\boldsymbol{\Phi}$ is called the *unified* constraint function, and it has to be an attractor. To stabilize it, $\dot{\boldsymbol{\Phi}} = \boldsymbol{u}_\Phi$ is taken into account. Picking $\boldsymbol{u}_\Phi = -\mathcal{P}_\Phi \boldsymbol{\Phi}$ as the simplest control law, it is obtained

$$\dot{\boldsymbol{\Phi}} + \mathcal{P}_\Phi \boldsymbol{\Phi} = 0 \Rightarrow \mathcal{W}\ddot{\boldsymbol{q}} + \dot{\mathcal{W}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{a}} + \mathcal{P}_\Phi \boldsymbol{\Phi} = 0 \quad (23)$$

in which $\mathcal{P}_\Phi = \mathcal{P}_\Phi^T > 0$ is an $(m + k) \times (m + k)$ real matrix. Substituting $\ddot{\boldsymbol{q}}$ from the model (3) in conjunction with (14) gives rise to an equation for the multiplier vector or the constraint force vector:

$$\mathcal{W}\mathcal{M}^{-1}\mathcal{W}^T \boldsymbol{\lambda} = \mathcal{W}\mathcal{M}^{-1}(\boldsymbol{\tau}_{ext} - \boldsymbol{n}) + \dot{\mathcal{W}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{a}} + \mathcal{P}_\Phi \boldsymbol{\Phi} \quad (24)$$

If $\mathcal{W}$ is full row rank, $\boldsymbol{\lambda}$ can be found and the forces are given by (14). If $\mathcal{W}$ is full column rank, the constraint forces can be directly computed by means of the generalized inverse (Moore-Penrose pseudo-inverse) of $\mathcal{W}$. In this case, due to the full column rank property of $\mathcal{W}$, the constraint forces are unique. The very least requirement for full rank Jacobian matrix is to avoid conflicting or redundant constraints. The result is formalized in Theorem 1.

Theorem 1. *The control law arising from* (24) *make the system* (3) *globally exponentially stable such that the constraint* (4) *and* (9) *hold if $\mathcal{W}$ is full (row or column) rank.*

Proof: (Sketch) Taking $V = \frac{1}{2}(\boldsymbol{G}^T\boldsymbol{G} + \boldsymbol{\Phi}^T\boldsymbol{\Phi})$ as the Lyapunov function and differentiating along the trajectories of (20) and (23), it is obtained $\dot{V} = -\boldsymbol{G}^T\mathcal{P}_G\boldsymbol{G} - \boldsymbol{\Phi}^T\mathcal{P}_\Phi\boldsymbol{\Phi}$. The existence of the control law was already discussed. ■

Remark 1. Basically, Method I and Method II are equivalent. However, Method II is preferable since it provides a systematic and step-by-step procedure by which dealing with modeling uncertainties and environmental disturbances is considerably easier. To counteract Δ, it is only required to consider $\dot{\boldsymbol{\Phi}} = \boldsymbol{u}_\Phi + \Delta$. Then $\boldsymbol{u}_\Phi$ is designed in a way that the control system becomes robust with respect to Δ. However, for Method I, this is unclear where Δ should be located and which control law is in charge of robustification.

IV. Path Maneuvering of Marine Surface Craft

In this section, the proposed method is utilized to design a path-following controller for a marine craft.

A. 3-DOF Marine Craft

A standard model for marine craft is considered [2]. The position and the heading of the craft are represented by $\boldsymbol{q} = [x, y, \psi]^T \in \mathbb{R}^2 \times [-\pi, \pi]$ in the inertial frame $\{i\}$. The body-fixed velocities are given by $\boldsymbol{\nu} = [u, v, r]^T \in \mathbb{R}^3$ which are related to the generalized velocities through the kinematic relation $\dot{\boldsymbol{q}} = \mathcal{R}(\psi)\boldsymbol{\nu}$ in which $\mathcal{R} = \mathrm{diag}(\mathcal{R}_1(\psi), 1)$ and

$$\mathcal{R}_1(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$$

The kinetics in the body-frame $\{b\}$ is described by

$$\mathcal{M}^b\dot{\boldsymbol{\nu}} + \mathcal{C}^b(\boldsymbol{\nu})\boldsymbol{\nu} + \mathcal{D}^b(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}^b + \boldsymbol{\tau}_c^b, \quad \boldsymbol{\tau}_c^b = \mathcal{R}^T(\psi)\boldsymbol{\tau}_c \quad (25)$$

where $\boldsymbol{\tau}^b$ and $\boldsymbol{\tau}_c^b$ denote the actuator force vector and the constraint force vector, expressed in $\{b\}$, respectively. In (25), $\mathcal{M}^b = \mathcal{M}^{b^T} > 0, \dot{\mathcal{M}}^b = 0$, is the mass and inertia matrix plus added mass; $\mathcal{C}^b(\boldsymbol{\nu})$ is the Coriolis and centripetal matrix, and $\mathcal{D}^b(\boldsymbol{\nu})$ is the matrix of hydrodynamic damping as

$$\mathcal{M}^b = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{bmatrix}, \mathcal{D}^b = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix} \quad (26a)$$

$$\mathcal{C}^b = \begin{bmatrix} 0 & 0 & -(m_{22}v + m_{23}r) \\ 0 & 0 & m_{11}u \\ m_{22}v + m_{23}r & -m_{11}u & 0 \end{bmatrix} \quad (26b)$$

Surge is assumed to be decoupled from sway and yaw. Representation of (25) in $\{i\}$ is (3) while for horizontal motion $\boldsymbol{g}(\boldsymbol{q}) = 0$. In the case of underactuation, $\tau_2 = 0$; i.e. $\boldsymbol{\tau}^b = [\tau_1, 0, \tau_3]^T$.

Remark 2. In [1], the constraint forces (14) are applied to the model of marine craft expressed in $\{b\}$. However, since constraints are defined in $\{i\}$, the respective forces are generalized forces and have to be expressed in $\{i\}$.

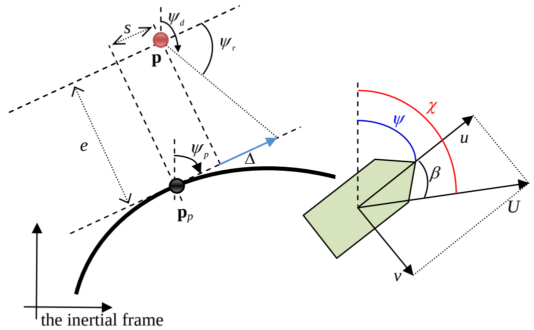

Fig. 2: The geometric representation of the problem

B. Problem Statement

Consider a geometric planar path parameterized by a scalar variable $\theta \in \mathbb{R}$. Each point on the path is denoted by $\boldsymbol{p}_p = [x_p(\theta), y_p(\theta)]^T \in \mathbb{R}^2, \forall\theta \in \mathbb{R}$. The path is smooth and geometrically feasible to be followed by the craft. It is conceived that a virtual vehicle whose generalized coordinate vector is denoted by $\boldsymbol{q}_p(t) = [x_p(\varpi), y_p(\varpi), \psi_p(\varpi)]^T, \forall\varpi(t) \in \mathbb{R}$ moves along the path, see Fig. 2. Certainly, its position is equal to that of the path at each instant of time, and the orientation is given by

$$\psi_p(\varpi) = \arctan(y_p^{\varpi}(\varpi)/x_p^{\varpi}(\varpi)) \tag{27}$$

where $x_p^{\varpi} \triangleq \frac{\partial x_p}{\partial \varpi}$. ψ_p is called the path tangential angle. The overall speed of the virtual craft denoted by U_p is found by

$$U_p = \dot{\varpi}\sqrt{(x_p^{\varpi})^2 + (y_p^{\varpi})^2} \Rightarrow \dot{\varpi} = \frac{U_p}{\sqrt{(x_p^{\varpi})^2 + (y_p^{\varpi})^2}} \tag{28}$$

This value has to be determined according to the objectives. If the position of the craft is denoted by $\boldsymbol{p} = [x, y]^T$, the error between the craft and the virtual reference represented in the body frame attached to the virtual reference is given by:

$$\boldsymbol{\varepsilon} = \mathcal{R}_1^T(\psi_p)(\boldsymbol{p} - \boldsymbol{p}_p(\varpi)) \tag{29}$$

The error vector ε consists of the along-track error s and the cross-track error e; i.e., $\boldsymbol{\varepsilon} = [s, e]^T$, see Fig. 2. The problem is stated as:

Path-following Problem Statement: We aim to force the craft to converge to and follow the path with the desired forward speed u_d.

Rephrased Problem Statement: The motion of the craft is restricted to the motion of the virtual reference. The distance error between them is desired to be zero, and the virtual reference moves with the desired speed.

To mathematically formulate the problem, a guidance rule proposed by [20] is used. The rule states that the relative angle ψ_r depends on the cross-track error and is given by:

$$\psi_r(e(t)) = \arctan(-e/\Delta) \tag{30}$$

where $\Delta > 0$ is referred to as the *lookahead distance* and specifies the point ahead of the direct projection of the craft to the tangent, toward which the craft is asked to move. Therefore, the desired heading angle is:

$$\psi_d(t) = \psi_r + \psi_p \tag{31}$$

Consequently, defining $\tilde{u} \triangleq u - u_d$, $\tilde{v} \triangleq v - v_d$, and $\tilde{\psi} \triangleq \psi - \psi_d$, the constraints are selected as

$$\boldsymbol{G} = \left[\tilde{\psi}, \varepsilon^T\right]^T \tag{32}$$

$$\boldsymbol{K} = [\tilde{u}, \tilde{v}]^T \equiv \left[\mathcal{R}_1^T(\psi), 0_{2\times1}\right]\dot{\boldsymbol{q}} - \left[\begin{array}{cc} u_d & v_d \end{array}\right]^T \tag{33}$$

C. Controller Design: Fully Actuated Marine Craft

We are going to find $\mathcal{W}_G$ and the unified constraint $\boldsymbol{\Phi}$ so as to use (24). Differentiating (29) and using the fact that $\dot{\mathcal{R}}_1(\alpha) = \mathcal{R}_1(\alpha)\mathcal{S}_1(\dot{\alpha})$ where

$$\mathcal{S}_1(\dot{\alpha}) = \left[\begin{array}{cc} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{array}\right]$$

one can find $\dot{\varepsilon}$. Considering $\mathcal{P}_G = \text{diag}(p_{G1}, \mathcal{P}_{G\varepsilon}), \mathcal{P}_{G\varepsilon} = \text{diag}(p_{G2}, p_{G3}), \mathcal{P}_{\Phi} = \text{diag}(p_{\Phi1}, p_{\Phi2}, p_{\Phi3}, \mathcal{P}_{\Phi\varepsilon})$, and $\mathcal{P}_{\Phi\varepsilon} = \text{diag}(p_{\Phi4}, p_{\Phi5})$, one may get

$$\mathcal{W}_G = \left[\begin{array}{cc} 0_{1\times2} & 1 \\ \mathcal{R}_1^T(\psi_p) & 0_{2\times1} \end{array}\right] \Rightarrow \mathcal{W} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathcal{R}^T(\psi)} \\ \mathcal{R}_1^T(\psi_p) & 0_{2\times1} \end{array}\right]$$

$$\boldsymbol{a} = \left[\begin{array}{c} -u_d \\ -v_d \\ -\dot{\psi}_d + p_{G1}\tilde{\psi} \\ \bar{\boldsymbol{a}} \end{array}\right], \bar{\boldsymbol{a}} = -\left[\begin{array}{c} U_p \\ 0 \end{array}\right] + S_1^T(\psi_p)\boldsymbol{\varepsilon} + \mathcal{P}_{G\varepsilon}\boldsymbol{\varepsilon}$$

Using the kinematics, (23) is transformed into

$$\mathcal{W}\mathcal{R}(\psi)\dot{\boldsymbol{\nu}} + \frac{d}{dt}(\mathcal{W}\mathcal{R}(\psi))\boldsymbol{\nu} + \dot{\boldsymbol{a}} + \mathcal{P}_{\Phi}\boldsymbol{\Phi} = 0 \tag{34}$$

If $\mathcal{J} \triangleq \mathcal{W}\mathcal{R}(\psi) \in \mathbb{R}^{5\times3}$, using (25), it follows that

$$\boldsymbol{\tau}_c^b = \mathcal{C}^b(\boldsymbol{\nu})\boldsymbol{\nu} + \mathcal{D}^b(\boldsymbol{\nu})\boldsymbol{\nu} - \mathcal{M}^b\mathcal{J}^{\dagger}\left(\dot{\mathcal{J}}\boldsymbol{\nu} + \dot{\boldsymbol{a}} + \mathcal{P}_{\Phi}\boldsymbol{\Phi}\right) \tag{35}$$

where $\mathcal{J}^{\dagger} = (\mathcal{J}^T\mathcal{J})^{-1}\mathcal{J}^T$; thus,

$$\mathcal{J}^{\dagger} = \left[\begin{array}{cc} \mathcal{I} & 0_{3\times2} \end{array}\right] + \mathcal{J}_{\varepsilon} \tag{36}$$

$\mathcal{J}_{\varepsilon}$ is easily obtained by inspection. Note that $\det(\mathcal{J}^T\mathcal{J}) = 4$ and the forces do exist. Also, notice that (35), leading to the constraint force expressed in $\{b\}$, is another form of (24). Assuming that all position and velocity variables are measured, there is no unknown term in (35) except U_p which can be selected in a way that the virtual reference moves with the desired speed of the craft; i.e. $U_p = \sqrt{u_d^2 + v_d^2}$ in which $u_{d,\max} > u_d > u_{d,\min} > 0$. A similar condition applies to v_d. Define $\boldsymbol{\epsilon} \triangleq [\tilde{u}, \tilde{v}, \dot{\tilde{\psi}}, \tilde{\psi}, \dot{e}, e, \dot{s}, s]^T$. The following result is the consequence of Theorem 1 for fully actuated marine craft.

Corollary 1. *The constraint force* (35) *renders the error* $\boldsymbol{\epsilon} = 0$ *GES if* ψ_r *is obtained by* (30) *and* ϖ *is updated by* (28).

Proof: It follows from Theorem 1. ■

D. Controller Design: Underactuated Marine Craft

In the case of undeactuation, the second component of $\boldsymbol{\tau}_c^b \triangleq [\tau_1, \tau_2, \tau_3]^T$ cannot be applied to the system. Moreover, τ_1 and τ_3 are functions of v_d. Thus, v_d must be a bounded signal so that the control signals are bounded. Taking v_d as a free signal and setting $\tau_2 = 0$, a differential equation is brought into play, consisting of the system dynamics and the controller structure. It is given by

$$\frac{1}{2}m_{22}\dot{v}_d = -n_{22}v_d + g(\boldsymbol{\epsilon}, t) + v_{v_d}(t) \tag{37}$$

in which $g(\boldsymbol{\epsilon}, t) = \frac{1}{2}m_{22}\sin(\gamma)\dot{u}_d + \boldsymbol{\Gamma}(\boldsymbol{\epsilon}, t)^T\dot{\boldsymbol{\epsilon}}$, and $v_{v_d}(t) = (m_{11}u_d + n_{23})\dot{\psi}_p + m_{23}\ddot{\psi}_p$, and

$$\begin{aligned}\boldsymbol{\Gamma} = [&m_{11}(\dot{\tilde{\psi}} + \dot{\psi}_p - \kappa\dot{e}), d_{22} - 0.5m_{22}p_{\Phi 2},\\ &d_{23} + m_{11}u_d - m_{23}p_{\psi 2} + 0.5m_{22}u, -m_{23}p_{\psi 1},\\ &-\kappa m_{11}u_d + m_{23}\kappa p_{e2}\dot{e} - m_{22}(\mathcal{H}_{\varepsilon 1}^{\{2,2\}} + 0.5\kappa u) - \kappa d_{23},\\ &2m_{23}\kappa^2\dot{e}^2/\Delta + \kappa m_{23}p_{e1} - m_{22}\mathcal{H}_{\varepsilon 1}^{\{2,2\}},\\ &-m_{22}\mathcal{H}_{\varepsilon 2}^{\{2,1\}}, -m_{22}\mathcal{H}_{\varepsilon 1}^{\{2,1\}}]^T\end{aligned}$$

where $\kappa = \frac{\Delta}{\Delta^2+e^2}$, and $\mathcal{A}^{\{i,j\}}$ denotes $\{i,j\}$-entry of $\mathcal{A}$, and $\mathcal{H}_{\varepsilon 2} \triangleq \frac{1}{2}\mathcal{R}(\gamma)\left(\mathcal{S}(\gamma)^T + \mathcal{P}_{\varepsilon 2}\right), \mathcal{H}_{\varepsilon 1} \triangleq \frac{1}{2}\mathcal{R}(\gamma)\left(\mathcal{S}(\dot{\gamma})^T + \mathcal{P}_{\varepsilon 1}\right)$ If $p_{e1} = (\mathcal{P}_{G\varepsilon}\mathcal{P}_{\Phi\varepsilon})^{\{2,2\}}$ and $p_{e2} = (\mathcal{P}_{G\varepsilon} + \mathcal{P}_{\Phi\varepsilon})^{\{2,2\}}$, then

$$\ddot{e} + p_{e2}\dot{e} + p_{e1}e = 0$$
$$\dot{\psi}_r = -\kappa\dot{e}, \qquad \ddot{\psi}_r = -\kappa\ddot{e} + 2\kappa^2\dot{e}^2 e/\Delta$$

which indicates that $\dot{\psi}_r$ and $\ddot{\psi}_r$ are globally bounded. The outcome is formalized in Corollary 2.

Corollary 2. *For underactuated marine surface craft, the constraint force* (35) *and differential equation* (37) *make the craft globally exponentially converge to the path while $\tilde{u} = 0$ is UGES, provided* (30) *and* (28) *are used. It also ensures that the unactuated sway velocity $v \in \mathcal{L}_\infty$.*

Proof: Based on Corollary 1, $\boldsymbol{\epsilon} = 0$ is UGES. Thus, path-maneuvering objectives are achieved. Furthermore, it is inferred that $\lim_{t\to\infty} v(t) = v_d(t)$. As explained, v_d is undefined a priori and τ_1, τ_3 depend on this signal. Providing it is proven $v_d \in \mathcal{L}_\infty$, it is concluded that $\tau_1, \tau_3, v \in \mathcal{L}_\infty$. The system $\frac{1}{2}m_{22}\dot{v}_d = -n_{22}v_d$ is GES and input-to-state stable [21] if $n_{22} > 0$. Looking at $g(\boldsymbol{\epsilon}, t) + v_{v_d}(t)$ as globally bounded input, v_d is globally bounded, accordingly. ■

Remark 3. In the underactuated case, we pick $U_p = u_d$.

At this point, it is helpful to clarify how *explicit* incorporation of the error $\boldsymbol{\varepsilon}$ influences the methodology and how it differs from other related articles. Using the decomposition (36), (35) can be regarded as $\boldsymbol{\tau}_c^b = \boldsymbol{\tau}_n + \boldsymbol{\tau}_\varepsilon$ in which

$$\boldsymbol{\tau}_n = \mathcal{C}^b(\boldsymbol{\nu})\boldsymbol{\nu} + \mathcal{D}^b(\boldsymbol{\nu})\boldsymbol{\nu} - \mathcal{M}^b[I_3, 0_{3\times 2}]\left(\dot{\mathcal{J}}\boldsymbol{\nu} + \dot{\boldsymbol{a}} + P_\Phi\boldsymbol{\Phi}\right)$$
$$\boldsymbol{\tau}_\varepsilon = -\mathcal{M}^b\mathcal{J}_\varepsilon\left(\dot{\mathcal{J}}\boldsymbol{\nu} + \dot{\boldsymbol{a}} + P_\Phi\boldsymbol{\Phi}\right)$$

$\boldsymbol{\tau}_n$ is the force that removes nonlinearity of the system and makes $\tilde{u}, \tilde{v}, \dot{\tilde{\psi}}, \tilde{\psi}$ GES while $\boldsymbol{\tau}_\varepsilon$ is the force due to the inclusion of $\boldsymbol{\varepsilon}$ to make it GES. It is apparent if $\boldsymbol{\varepsilon} \neq 0$, $\boldsymbol{\tau}_\varepsilon$ affects $\boldsymbol{\tau}_c^b$. Thus, not only the heading angle is exploited to make the craft converge to the path, but also the velocity components are manipulated in a way that $\boldsymbol{\varepsilon}$ decreases to zero exponentially fast. This feature is useful when the speed assignment task can be occasionally sacrificed for the geometric task. Indeed, many articles [20], [22], [23] in this field do not take advantage of speed for fast convergence. They usually use a feedback linearization loop with PI-type controller, leading to a perfect speed assignment in ideal circumstances. It is worth mentioning that they stabilize the along-track error by means of the virtual reference speed, U_p, as an auxiliary input. The proposed approach is also different in this regard.

V. Simulation Results

A model of Cyber Ship 2, which is $L = 1.3$m long, is selected. The system parameters based on (26) are

$$m_{11} = 22.8, m_{22} = 33.8, m_{23} = m_{32} = 1.01, m_{33} = 2.76$$
$$n_{11} = 2, n_{22} = 7, n_{23} = n_{32} = 0.1, n_{33} = 0.5$$

The ship is unactuated in sway. The desired forward speed is $u_d = 0.1$m/s. The maximum surge force and the yaw moment are 2N and 1.5Nm, respectively. Saturation blocks are placed in the simulation model, and the controller gains are picked so as to avoid saturation while reasonable growth rate of forces and acceptable performance are accomplished. The lookahead distance is selected as $\Delta = 1$m. The controller gains are picked as $p_{\Phi 1} = p_{\Phi 2} = 30$ and the other gains in $\mathcal{P}_{2G}$ and $\mathcal{P}_\Phi$ are equal to 2. In the following, two paths are considered. One is a sinus-like path, described by $\boldsymbol{p}_1(\theta) = [\theta, 10\sin(\theta/2)]^T$. The initial conditions are $\boldsymbol{q}_0 = [-1, -1, \pi/2]^T$ and $\boldsymbol{\nu}_0 = [0.05, 0, 0]^T$. The results are shown in Figs. 3 & 4. The second path is an ellipsoid expressed as $\boldsymbol{p}_2(\theta) = [5\cos(\theta), 4\sin(\theta)]^T$. The craft is initially located at $\boldsymbol{q}_0 = [5, -4, \pi/2]^T$. Fig. 5 shows the result. We compare the proposed controller with one typical controller for curved path following presented in [20]. In the typical controller, the speed of the virtual reference is used to stabilize the along-track error s and convergence to the path is supposed to be achieved by the guidance law (30). The controller gains of the typical controller are kept the same as those of the proposed controller.

Discussion

The path $\boldsymbol{p}_1$ is a sinusoid path with sharp turns. As demonstrated by Fig. 3, the proposed method outperforms at sharp turning points. The reason is manipulation of the forward velocity to reach the path faster and it is the consequence of the explicit inclusion of $\boldsymbol{\varepsilon}$ in the design. However, the adjustment may cause reverse thrust, which is impractical in marine craft. To circumvent the problem, the controller gains for the velocity loops are advised to be selected high. As seen in Fig. 4, the heading actions are similar in both controllers; then, the influential difference is the forward velocity. The typical controller leads the craft to move with the constant speed 0.1m/s at all instants of time.

In Fig. 5, the ellipsoid $\boldsymbol{p}_2$ is the desired path. The objective is to see the effect of the guidance parameter Δ. It is apparent that smaller Δ results in better convergence. The reason is that the ship always points towards a point on the tangent to the path rather than a point on the path. Therefore, larger Δ leads to a larger offset to the path.

VI. Conclusions

In this paper, a methodology for motion control of mechanical systems is developed. Inspired by analytical mechanics, the control problem is treated as a constrained multi-body system; then, techniques for forward dynamics simulation is used for the stabilization task and the control law is derived. The method is employed for path maneuvering of marine surface craft, which leads to a novel controller that manipulates the speed for fast convergence to the path. This feature is a result of explicit inclusion of geometric distance ε in the design and can be of interest in exact path-following missions.

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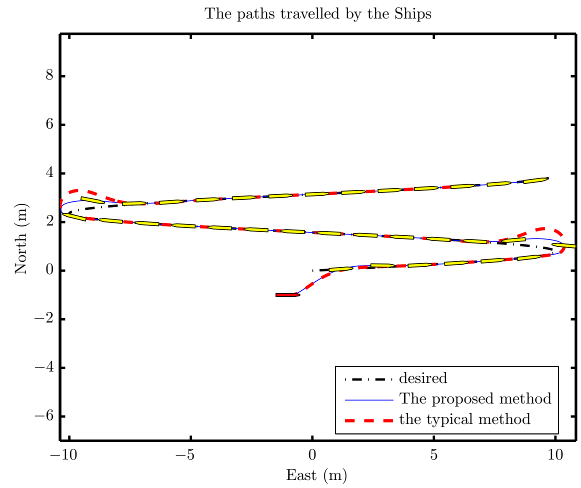

Fig. 3: Comparison between the proposed method and a typical method for the path $\boldsymbol{p}_1$.

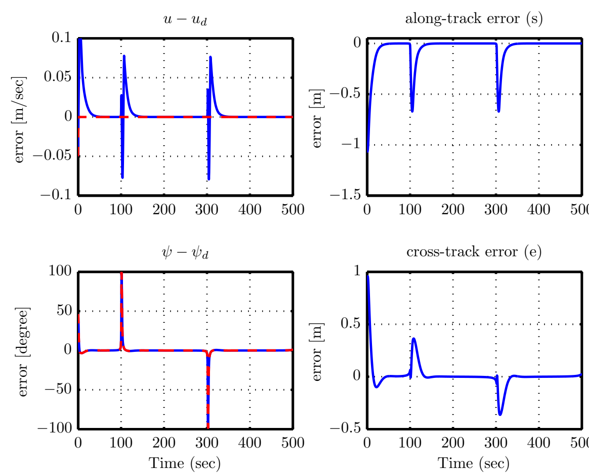

Fig. 4: Control objectives related to Fig. 3. The blue solid line belongs to the proposed controller.

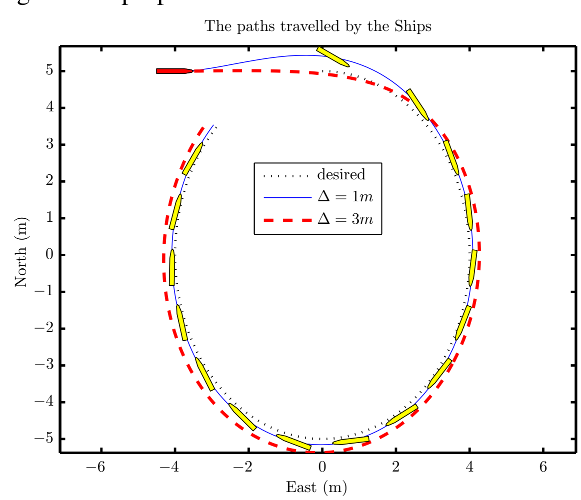

Fig. 5: Following the ellipsoid $\boldsymbol{p}_2$.