A Nonlinear 7-DOF Model for U-Tanks of Arbitrary Shape

Christian Holden[a,*], Thor I. Fossen[a,b]

[a] *Centre for Ships and Ocean Structures, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*

[b] *Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*

Abstract

This work presents a novel nonlinear 7-DOF model for ships equipped with u-tanks of arbitrary shape. The model uses the standard six degrees of freedom for the ship, in addition to a single degree of freedom for the tank fluid. The ship–tank interaction was modeled with Hamiltonian (analytical) mechanics, end external forces (such as those due to the surrounding ocean, actuator forces and various damping forces) were added later in a Newtonian framework. These external forces were not explicitly modeled in this work. The model was compared to two (significantly simpler and less powerful) models in literature, one of which was experimentally verified. Under the same assumptions, the new model is identical to the experimentally verified one, and contains several effects not found in the other.

Keywords:
u-tank, anti-roll tank, u-tube, analytical mechanics, Hamiltonian mechanics

1. Introduction

The rolling motion of ships can be quite dangerous (Beck et al., 1989; Faltinsen, 1998; Fossen, 2011; Lloyd, 1998). As such, there is a necessity to reduce this unwanted motion. Unfortunately, the actuators used to move the ship are unsuitable to control roll (with the exception of rudder-roll-damping) (Fossen, 2011; Perez, 2005). Therefore, specialized control systems will have to be used.

There are several types of roll control devices. These include fins, gyro stabilizers, flume tanks and u-tanks (see Figure 2) (Perez, 2005). These devices have intrinsic strengths and weaknesses. Fins are external and potentially vulnerable, increase drag, and the effectiveness is usually greater at high speeds. On the other hand, they have fast response times and are easy to control. U-tanks (also known as u-tube anti-roll tanks, u-shaped anti-roll tanks) are internal, and thus have no drag or vulnerability penalties, and are equally effective at low and high speeds. The downsides are that they are more complicated to model and control, can take up valuable space inside the hull and – most problematically – will have adverse effects in certain conditions.

A u-tank operates on the following principle: As the ship rolls, the fluid in the u-tank (usually water, but any liquid could be used) moves with it. For a passive tank, the fluid should move with

*Corresponding author. *Tel:* +47 99 61 88 81. *Fax:* +47 73 59 45 99
Email addresses: `c.holden@ieee.org` (Christian Holden), `fossen@ieee.org` (Thor I. Fossen)
URL: `kybernetes.net` (Christian Holden), `www.itk.ntnu.no/ansatte/Fossen_Thor` (Thor I. Fossen)

the same frequency as the rolling motion, but lagging a quarter of a period behind (Lloyd, 1998). The vessel's kinetic energy is transformed into kinetic and potential energies of the tank fluid. Part of this energy is then dissipated by damping effects in the tank, such as vortex shedding and fluid viscous effects related to skin friction on the walls of the tank (Beck et al., 1989).

For a passive tank, both the ship and the tank fluid will move with the frequency of the waves (Beck et al., 1989; Fossen, 2011). This frequency will depend on the sea state, but the response of the ship will be much more severe if the exciting moment is at the ship's natural roll frequency (Nayfeh and Mook, 1995). Therefore, the u-tank should be constructed so that it is most effective at damping the roll motion at this frequency, and a u-tank should thus be designed to have its natural frequency approximately that of the ship's natural roll frequency (Faltinsen and Timokha, 2009; Lloyd, 1998).

The natural roll frequency will depend on loading conditions and other factors, and does not necessarily take the design value. The exact value might be unknown. The natural frequency of the u-tank fluid can be changed by changing its design or adding or removing fluid (Lloyd, 1998). Unfortunately, the natural frequency has very little sensitivity to changes in the fluid level, rendering it impractical to change the tank's natural frequency after it has been installed. Furthermore, a u-tank will have a limited range of exciting frequencies in which it is effective (Faltinsen and Timokha, 2009; Lloyd, 1998). For other frequencies, it may increase rather than decrease roll, and an incorrectly tuned u-tank can therefore cause more problems than it will solve. An incorrectly tuned u-tank may also simply have negligible effect on roll. Nevertheless, u-tanks are still in use throughout the world (Marzouk and Nayfeh, 2009; Moaleji and Greig, 2007; Perez, 2002, 2005).

The problems with a passive u-tank can be solved by adding an active control system. Pressurized air, pumps or simply a series of valves could be used to control the motion of the tank fluid. Ideally, a powerful and accurate model should be used to design and test such a system.

Several u-tank models exist in literature. One of the oldest (Moaleji and Greig, 2007) is that of Goodrich (1968). Kagawa et al. (1989) presented a u-tank model for the purpose of reducing unwanted sway motion in skyscrapers. The most commonly used model is that of Lloyd (1989, 1998). More recently, u-tank models have appeared in Faltinsen and Timokha (2009), Marzouk and Nayfeh (2009), Neves et al. (2009), Holden et al. (2011) and Holden (2011).

The existing models have several limitations. Chief among these are that they are derived for rectangular-prism u-tanks (i.e., u-tanks consisting of three rectangular boxes) (Holden et al., 2011; Kagawa et al., 1989; Lloyd, 1989, 1998; Marzouk and Nayfeh, 2009; Moaleji and Greig, 2007; Neves et al., 2009; Sellars and Martin, 1992)[1], despite the fact that several actually installed tanks do not match this shape (Perez, 2002, 2005; Sellars and Martin, 1992). A model valid also for more generic shapes is therefore likely to be useful. Furthermore, most existing models are linear (Frahm, 1911; Goodrich, 1968; Kagawa et al., 1989; Lloyd, 1989, 1998; Moaleji and Greig, 2007), and technically only valid for low-amplitude motions. Finally, they are often limited in degrees of freedom, typically to two

[1]Technically, the model in Marzouk and Nayfeh (2009) is for a tank consisting of three cylinders, but this is functionally identical to a rectangular-prism u-tank.

(usually roll and a tank state) (Frahm, 1911; Goodrich, 1968; Holden et al., 2011; Kagawa et al., 1989) or four (usually sway, roll, yaw and a tank state) (Faltinsen and Timokha, 2009; Lloyd, 1989, 1998).

It is known that the modes most affected by/affecting a u-tank are sway, roll and yaw, and that a 2-DOF model might be too low order for a thorough analysis (Faltinsen and Timokha, 2009; Lloyd, 1998). However, all degrees of freedom will be affected by and affect a u-tank, and a high-order model can always easily be reduced to a low-order model, but not the other way around.

In this work, we therefore present a novel 7-DOF nonlinear model for ships equipped with a u-tank of arbitrary shape. The ship–tank interaction is modeled with Hamiltonian (analytical) mechanics[2], and the forces and moments of the surrounding ocean added later in a Newtonian framework (these latter forces are not explicitly modeled in this work). The novel model can accurately model u-tanks of arbitrary shape, describe high-amplitude motion and has seven degrees of freedom (the well-known six of the ship and a single tank state). The main disadvantage of the new model is that it is quite complex. Simplifications can, however, be easily made. Furthermore, the motion of the tank fluid is assumed to be one-dimensional and as a consequence does not accurately model the behavior of the surface motion of the tank fluid.

The model is compared to the commonly used (linear, 4-DOF, rectangular-prism u-tank) model of Lloyd (1989, 1998) and the experimentally verified (nonlinear, 2-DOF, rectangular-prism u-tank) model of Holden et al. (2011). Under the same assumptions as the existing models, the novel model is auxiliary perfect match for that of Holden et al. (2011), and incorporates several effects left out of the model of Lloyd (1989, 1998).

The rest of the article is organized as follows: Section 2 presents a brief nomenclature. Section 3 defines the reference frames and kinematic representation used in this work. Section 4 presents the main assumptions that form the basis of the model derivation, and properly defines the u-tank and the state which describes the motion of the u-tank fluid. Section 5 presents a Hamiltonian model of the ship–tank system (excluding the effects of the surrounding ocean). Section 6 adds the effects of forces not presented in Section 5, such as those of the surrounding ocean, actuators and dissipative effects. Section 7 compares the novel model to those of Lloyd (1989, 1998) and Holden et al. (2011). Section 8 presents the conclusions. In the appendices are found auxiliary results and derivations, to wit: Derivation of the potential energy (Appendix A), derivation of the kinetic energy (Appendix B), properties of certain matrices needed in the proofs (Appendix C), a definition of a virtual work principle used in deriving the dynamics (Appendix D) and the derivation of the Hamiltonian dynamics (Appendix E). The references are found at the very end of this article.

2. Nomenclature

This section lists the variables used in this work.

[2] Most other models have been derived by Newtonian mechanics (Faltinsen and Timokha, 2009; Frahm, 1911; Goodrich, 1968; Lloyd, 1989, 1998; Marzouk and Nayfeh, 2009; Moaleji and Greig, 2007; Neves et al., 2009; Sellars and Martin, 1992), but both approaches result in the same dynamic equations under the same assumptions.

In general, matrices will be written in uppercase with italic typeface, e.g., A. Vectors and scalars are typically written in lowercase with italic typeface, e.g., a. Whether it is a vector or scalar will be stated in the text, but should largely be clear from context.

If the vector has an interpretation as a point, velocity or angular velocity in physical $\mathbb{R}^3$, a superscript will typically denote which reference frame is used to describe the vector, e.g., r^n would denote that r is given in the n-frame. Only two frames are used, the b-frame (fixed to the ship) and the n-frame (fixed to the Earth and considered inertial), see Section 3.

$\mathbb{I}_n \in \mathbb{R}^{n\times n}$: The n-by-n identity matrix.

$0_{m\times n} \in \mathbb{R}^{m\times n}$: The m-by-n zero matrix.

$e_z = [0,0,1]^\top$: Unit vector in the z-direction (in $\mathbb{R}^3$).

$S(\cdot) \in \mathrm{SS}(3) \subset \mathbb{R}^{3\times 3}$: A skew-symmetric matrix representing the cross-product in $\mathbb{R}^3$. $S(x)y = x \times y \,\forall\, x, y \in \mathbb{R}^3$.

$R \in \mathrm{SO}(3) \subset \mathbb{R}^{3\times 3}$: Rotation matrix representing the orientation of the b-frame relative to the n-frame. If r is a vector in physical $\mathbb{R}^3$, then $r^n = Rr^b$.

$g > 0 \in \mathbb{R}$: The acceleration of gravity.

$x^n = [x, y, z]^\top \in \mathbb{R}^3$: The position of the origin of the b-frame, described in the n-frame.

$\eta = [\eta_r, \eta_i^\top]^\top \in \mathbb{R}^4$: Quaternion describing the orientation of the b-frame relative to the n-frame. $\eta_r = \mathrm{Re}\,(\eta) \in \mathbb{R}$, $\eta_i = \mathrm{Im}\,(\eta) \in \mathbb{R}^3$.

$q_t \in \mathbb{R}$: Generalized position of the tank fluid.

$q = [x^{n\top}, \eta^\top, q_t]^\top \in \mathbb{R}^8$: Generalized position.

$v^b \in \mathbb{R}^3$: The velocity of the b-frame relative to the n-frame, described in the b-frame.

$\omega^b \in \mathbb{R}^3$: The angular velocity of the b-frame relative to the n-frame, described in the b-frame.

$\nu = [v^{n\top}, \omega^{b\top}, \dot{q}_t]^\top \in \mathbb{R}^7$: Generalized velocity.

$G(\eta) \in \mathbb{R}^{3\times 4}$: Matrix relating $\dot{\eta}$ to ω^b; $\dot{\eta} = \frac{1}{2}G^\top(\eta)\omega^b$.

$\mathcal{P}(\eta) \in \mathbb{R}^{7\times 8}$: Matrix relating $\dot{q}$ to ν; $\dot{q} = \mathcal{P}^\top(\eta)\nu$.

ϕ, θ, ψ: Roll–pitch–yaw Euler angles.

$m > 0 \in \mathbb{R}$: Mass of the ship.

$I = I^\top > 0 \in \mathbb{R}^{3\times 3}$: The ship's moment of inertia tensor, in the body frame.

$r_g^b = [x_g^b, 0, z_g^b]^\top$: Position of the ship's center of gravity (excluding tank fluid), in the b-frame.

$\sigma \in \mathbb{R}$: Parameter describing the geometry of the tank.

$r_t^b(\sigma) = [x_t^b, y_t^b(\sigma), z_t^b(\sigma)]^\top \in \mathbb{R}^3$**:** A function describing the centerline of the u-tank, in the b-frame.

$A(\sigma) > 0 \in \mathbb{R}$**:** Cross-sectional area of the tank.

$\rho_t > 0 \in \mathbb{R}$**:** Density of the tank fluid.

$\varsigma_0 > 0 \in \mathbb{R}$**:** Mean level of tank fluid.

$\varsigma_p \in \mathbb{R}$**:** Instantaneous position of the tank fluid surface in the port side reservoir.

$\varsigma_s \in \mathbb{R}$**:** Instantaneous position of the tank fluid surface in the starboard side reservoir.

$A_0 > 0 \in \mathbb{R}$**:** An arbitrary parameter.

$M(q_t) \in \mathbb{R}^{7\times 7}$**:** Total, generalized inertia matrix of a ship with a u-tank.

$C(q_t, \nu) = -C^\top(q_t, \nu) \in \mathbb{R}^{7\times 7}$**:** Total Coriolis/centripetal matrix of a ship with a u-tank.

$D(\nu) \in \mathbb{R}^{7\times 7}$**:** Total damping matrix of a ship with a u-tank.

$k(q, t) \in \mathbb{R}^7$**:** Vector of generalized restoring forces on a ship with a u-tank.

$\tau_e \in \mathbb{R}^7$**:** Unmodeled generalized forces on a ship with a u-tank.

$\tau_c \in \mathbb{R}^7$**:** Generalized control forces on the ship and the u-tank.

3. Reference frames and kinematics

In this work, we will be working with two reference frames: One fixed to the ship (the "body" or "b" frame) and one fixed to the mean surface of the ocean (the "n" frame). The n frame is considered inertial by neglecting the Earth's motion.

For simplicity, we assume that the body frame is fixed at the transversal geometric center of the ship, with the x-axis pointing forwards, the y-axis starboard, and the z-axis down. We assume that the inertial frame is arranged so that the z-axis points with the gravity field and that the xy-plane is normal to the gravity field. Furthermore, the mean ocean surface is assumed to be an infinite, flat plane (there may be waves). See Figure 1.

A vector $r \in \mathbb{R}^3$ will be written as r^n in the inertial frame and r^b in the body frame. No superscript is given if $r \notin \mathbb{R}^3$, $r^n = r^b$ or r should not be interpreted as a vector in physical $\mathbb{R}^3$.

There are several ways to describe the rotation between the b-frame and the n-frame. Among those commonly used are the roll–pitch–yaw Euler angles and unit quaternions (also known as Euler parameters). There are several well-known issues with both of these representations (and with most alternatives).

Euler angles is a three-parameter representation. It suffers from a singularity: for certain orientations, the Euler angles are not well-defined. Furthermore, an infinite number of Euler angle triples are associated with each physical orientation (and rotation matrix): Let $\Theta \in \mathbb{R}^3$ be an Euler angle triple. Then, all $\Theta + 2\pi[k, l, m]^\top \;\forall\; k, l, m \in \{0, \pm 1, \pm 2, \ldots\}$ are distinct Euler angle triples (distinct points in phase space), but have the same physical interpretation and rotation matrix (i.e., an infinite mapping of possible rotations).

Unit quaternions is a four-parameter representation. It can be thought of as all vectors on the surface of the unit sphere in $\mathbb{R}^4$, with some extra operators. It does not have the singularity of the Euler angles, but does suffer from a double mapping of possible orientations: If η is a unit quaternion, then η and $-\eta$ have the same physical interpretation.

In other words, any rotation has two associated quaternions, and an infinite number of roll–pitch–yaw triples.

To avoid the singularity inherent in Euler angle representations and to avoid the complexities of more complex representations, we will use unit quaternions in this work.

First, we need to define the skew-symmetry operator. For any two vectors $x, y \in \mathbb{R}^3$, we define the cross-product matrix S so that

$$S(x)y = x \times y. \tag{1}$$

From this definition it follows that if $x = [x_1, x_2, x_3]^\top$, then

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times3}. \tag{2}$$

The position of the b-frame origin is related to the velocity of the b-frame by

$$\dot{x}^n = Rv^b \tag{3}$$

where R is a rotation matrix associated with the rotation of the b-frame relative to the n-frame.

We let $\eta = [\eta_r, \eta_i^\top]^\top \in \mathbb{R}^4$ with $\eta_r \in \mathbb{R}, \eta_i \in \mathbb{R}^3$ be the unit quaternion associated with the rotation of the b-frame relative to the n-frame. We define the Re and Im operators so that $\eta_r = \mathrm{Re}\,(\eta)$ and $\eta_i = \mathrm{Im}\,(\eta)$. Furthermore, $\eta^\top \eta = 1$.

The kinematics then satisfy certain properties:

Property 1. *If ω^b is the angular velocity of the b-frame relative to the n-frame in the b-frame, then*

$$\dot{R} = RS(\omega^b). \tag{4}$$

Proof. See Egeland and Gravdahl (2002); Fossen (2011). □

Property 2. *The rotation matrix R associated with the rotation of the b-frame relative to the n-frame can be written as*

$$R = EG^\top \tag{5}$$

where

$$E \triangleq \begin{bmatrix} -\eta_i & \eta_r \mathbb{I}_3 + S(\eta_i) \end{bmatrix} \in \mathbb{R}^{3\times4} \tag{6}$$

$$G \triangleq \begin{bmatrix} -\eta_i & \eta_r \mathbb{I}_3 - S(\eta_i) \end{bmatrix} \in \mathbb{R}^{3\times4}. \tag{7}$$

Alternate representations of the rotation matrix are

$$
\begin{aligned}
R &= \mathbb{I}_3 + 2\eta_r S(\eta_i) + 2S^2(\eta_i) && (8)\\
&= (\eta_r^2 - \eta_i^\top \eta_i)\mathbb{I}_3 + 2\eta_i\eta_i^\top + 2\eta_r S(\eta_i) && (9)\\
&= (2\eta_r^2 - 1)\mathbb{I}_3 + 2\eta_i\eta_i^\top + 2\eta_r S(\eta_i) && (10)\\
&= (1 - 2\eta_i^\top \eta_i)\mathbb{I}_3 + 2\eta_i\eta_i^\top + 2\eta_r S(\eta_i). && (11)
\end{aligned}
$$

Proof. See Egeland and Gravdahl (2002); Shivarama (2002); Shivarama and Fahrenthold (2004). □

Property 3. *The matrix G satisfies*

$$GG^\top = \mathbb{I}_3. \tag{12}$$

Proof. From the definition (7) of G, we have

$$
\begin{aligned}
GG^\top &= \begin{bmatrix} -\eta_i & \eta_r\mathbb{I}_3 - S(\eta_i) \end{bmatrix} \begin{bmatrix} -\eta_i^\top \\ \eta_r\mathbb{I}_3 + S(\eta_i) \end{bmatrix} = \eta_i\eta_i^\top + \eta_r^2\mathbb{I}_3 - S^2(\eta_i)\\
&= \eta_i\eta_i^\top + \eta_r^2\mathbb{I}_3 - \eta_i\eta_i^\top + \eta_i^\top\eta_i\mathbb{I}_3 = \eta^\top\eta\mathbb{I}_3 = \mathbb{I}_3 && (13)
\end{aligned}
$$

where it has been used that $\eta_r^2 + \eta_i^\top\eta_i = \eta^\top\eta = 1$. □

Property 4. *The matrix G satisfies*

$$G\eta = 0. \tag{14}$$

Proof. From the definition (7) of G we have,

$$G\eta = \begin{bmatrix} -\eta_i & \eta_r\mathbb{I}_3 - S(\eta_i) \end{bmatrix} \begin{bmatrix} \eta_r \\ \eta_i \end{bmatrix} = -\eta_r\eta_i + \eta_r\eta_i - S(\eta_i)\eta_i = 0 \tag{15}$$

where it has been used that $S(x)x = 0 \;\forall\; x \in \mathbb{R}^3$. □

Property 5. *The matrix G satisfies*

$$G\dot{\eta} = -\dot{G}\eta. \tag{16}$$

Proof. From Property 4 we have $G\eta = 0$. Differentiating on both sides with respect to time gives $\dot{G}\eta + G\dot{\eta} = 0 \;\Rightarrow\; \dot{G}\eta = -G\dot{\eta}$. □

Property 6. *The body-fixed angular velocity ω^b and the quaternion η are related by*

$$
\begin{aligned}
\dot{\eta} &= \frac{1}{2}G^\top\omega^b && (17)\\
\omega^b &= 2G\dot{\eta} = -2\dot{G}\eta. && (18)
\end{aligned}
$$

Proof. See Egeland and Gravdahl (2002); Shivarama (2002); Shivarama and Fahrenthold (2004). □

Property 7. *The matrix G and the body-fixed angular velocity ω^b are related by*

$$S(\omega^b) = 2G\dot{G}^\top = -2\dot{G}G^\top. \tag{19}$$

Proof. See Shivarama (2002); Shivarama and Fahrenthold (2004). □

4. Modeling hypothesis

A u-tank is simply two reservoirs of water or another liquid, one on the port side and the other at starboard, with a duct in between to allow the passage of liquid. To be able to model this intrinsically complicated behavior, some assumptions have to be made:

Assumption 1. The surface of the fluid in the tank is perpendicular to the centerline of the tank.

Assumption 2. The fluid in the tank is incompressible.

Assumption 3. The flow of fluid in the tank is one-dimensional.

Assumption 4. Tank fluid memory effects are negligible.

Assumption 5. The u-tank is placed at the transversal geometrical center of the ship.

Assumption 6. The tank is port–starboard symmetrical.

Assumption 7. There are no air bubbles in the tank.

Assumption 8. The centerline of the tank is smooth.

Assumption 9. The centerline of the tank runs from port to starboard.

Assumption 10. The ship's center of gravity (excluding the tank fluid) is given by $r_g^b = [x_g^b, 0, z_g^b]^\top \in \mathbb{R}^3$ and is constant.

Assumption 1 is clearly false for a ship in motion. Steady-state, the fluid surface would be horizontal due to the effects of gravity. In rough seas, in non-steady-state conditions, the actual fluid surface in the tank is likely to behave in a complicated and unpredictable fashion. Modeling this accurately without resorting to computational fluid dynamics is unfeasible. Assuming the fluid surface to be horizontal (a quasi-steady model of the tank fluid surface) would render the model significantly more complex and, in all likelihood, not be much more accurate than Assumption 1, as the fluid surface in unlikely to have time to settle down. Furthermore, Assumption 1 was also made in Holden et al. (2011), and the experimental results presented in that work indicate that this assumption does not result in any significant loss of accuracy.

Assumptions 1–9 imply that the tank fluid is parametrizable as a tube of varying cross-sectional area. Representing the centerline of the tube of fluid by $r_t^b(\sigma)$ with parameter σ, r_t^b can be written as

$$r_t^b(\sigma) = \left[x_t^b, y_t^b(\sigma), z_t^b(\sigma)\right]^\top . \tag{20}$$

The parameter σ is defined to have zero point at the ship centerline and be positive to port. The fluid surfaces are located at $\sigma = -\varsigma_s \leq 0$ (starboard side) and $\sigma = \varsigma_p \geq 0$ (port side). Thus, $\sigma \in [-\varsigma_s, \varsigma_p] \subset \mathbb{R}$ defines the fluid-filled part of the tank. When the fluid level is equal in both starboard and port side reservoirs, $\varsigma_p = \varsigma_s = \varsigma_0$, and $\sigma \in [-\varsigma_0, \varsigma_0] \subset \mathbb{R}$ defines the fluid-filled part of the tank.

Property 8 (Properties of r_t^b)**.** *r_t^b satisfies the following properties:*

- *x_t^b is a constant, per Assumption 9.*
- *The functions y_t^b and z_t^b are smooth (specifically, C^1 or greater), per Assumption 8.*
- *y_t^b is odd and lies in the second and fourth quadrant (i.e., $y_t^b(-\sigma) = -y_t^b(\sigma)$, $y_t^b(0) = 0$ and $y_t^b(\sigma) < 0 \,\forall\, \sigma > 0$), per Assumptions 5 and 6.*
- *z_t^b is even (i.e., $z_t^b(-\sigma) = z_t^b(\sigma)$), per Assumption 6.*
- $\max z_t^b = z_t^b(0)$*, per Assumptions 5 and 6.*

To fully describe the tank fluid, the cross-sectional area $A(\sigma)$ is also needed.

Property 9 (Properties of A)**.** *By Assumption 7, the fluid fills the entire area $A(\sigma) \,\forall\, \sigma \in [-\varsigma_s, \varsigma_p]$. Assumption 6 implies that $A(-\sigma) = A(\sigma) > 0$.*

See Figure 2 for an illustration of the u-tank and its parameters.

The chief physically measurable states of the system are the tank fluid levels ς_p, ς_s and the volumetric flow of the tank fluid Q (defined positive to port). ς_p and ς_s are related to the flow by

$$\dot{\varsigma}_p = \frac{Q}{A(\varsigma_p)}\,, \qquad \dot{\varsigma}_s = -\frac{Q}{A(\varsigma_s)}.$$

We define the generalized tank coordinate q_t as

$$q_t \triangleq \frac{1}{A_0} \int_{\varsigma_0}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma \tag{21}$$

where A_0 is an arbitrary constant with unit m^2.

We note that the total fluid volume in the tank, V_t, is constant (Assumption 2). Thus,

$$\begin{aligned} V_t &\triangleq \int_{-\varsigma_0}^{\varsigma_0} A(\sigma)\,\mathrm{d}\sigma = \int_{-\varsigma_s}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma = \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + \int_{-\varsigma_0}^{\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + \int_{\varsigma_0}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma \\ &= \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + V_t + A_0 q_t, \end{aligned}$$

This gives

$$q_t = -\frac{1}{A_0} \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma. \tag{22}$$

The time derivative of q_t is given by

$$\dot{q}_t = \frac{1}{A_0} A(\varsigma_p) \frac{\mathrm{d}\varsigma_p}{\mathrm{d}t} = \frac{Q}{A_0}. \tag{23}$$

By differentiating on both sides of (21) and (22) with respect to q_t, we get

$$\frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t} = \frac{A_0}{A(\varsigma_p)}\,, \qquad \frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t} = -\frac{A_0}{A(\varsigma_s)}. \tag{24}$$

We note that ς_p and ς_s can be written as functions of q_t. However, an analytical expression might not be possible to find. They are then implicitly defined by the relationships (21) and (22).

The speed of the tank fluid relative to the tank walls (i.e., the ship), at any point σ in the tank, is given by

$$\|v^b_{t,r}(\sigma, \dot{q}_t)\| = \frac{Q}{A(\sigma)} = \frac{A_0 \dot{q}_t}{A(\sigma)}.$$

From calculus, we know that velocity is tangential to the path, giving

$$v^b_{t,r}(\sigma, \dot{q}_t) = \frac{A_0 \dot{q}_t}{A(\sigma)} \frac{\mathrm{d}\bar{r}^b_t}{\mathrm{d}\sigma}(\sigma), \tag{25}$$

where

$$\frac{\mathrm{d}\bar{r}^b_t}{\mathrm{d}\sigma} \triangleq \frac{\frac{\mathrm{d}r^b_t}{\mathrm{d}\sigma}}{\left\|\frac{\mathrm{d}r^b_t}{\mathrm{d}\sigma}\right\|}. \tag{26}$$

Noting that $\mathrm{d}x^b_t/\mathrm{d}\sigma = 0$, we define

$$\frac{\mathrm{d}\bar{y}^b_t}{\mathrm{d}\sigma} \triangleq [0,1,0]\frac{\mathrm{d}\bar{r}^b_t}{\mathrm{d}\sigma} = \frac{\frac{\mathrm{d}y^b_t}{\mathrm{d}\sigma}}{\sqrt{\left(\frac{\mathrm{d}y^b_t}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}z^b_t}{\mathrm{d}\sigma}\right)^2}} \tag{27}$$

$$\frac{\mathrm{d}\bar{z}^b_t}{\mathrm{d}\sigma} \triangleq [0,0,1]\frac{\mathrm{d}\bar{r}^b_t}{\mathrm{d}\sigma} = \frac{\frac{\mathrm{d}z^b_t}{\mathrm{d}\sigma}}{\sqrt{\left(\frac{\mathrm{d}y^b_t}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}z^b_t}{\mathrm{d}\sigma}\right)^2}} \tag{28}$$

such that

$$\left(\frac{\mathrm{d}\bar{y}^b_t}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}\bar{z}^b_t}{\mathrm{d}\sigma}\right)^2 \equiv 1.$$

Of course, the ship (and the tank with it) is translating with velocity v^b and rotating with angular velocity ω^b relative to the inertial frame. Thus, the velocity of the tank fluid relative to the inertial frame, at any point σ in the tank, is

$$v^b_t(\sigma, \dot{q}_t, v, \omega) = v^b + \omega^b \times r^b_t(\sigma) + \frac{A_0 \dot{q}_t}{A(\sigma)} \frac{\mathrm{d}\bar{r}^b_t}{\mathrm{d}\sigma}(\sigma). \tag{29}$$

5. Hamilton's equations for the ship–tank system

To use analytical mechanics, we need to know the potential and kinetic energies of the system. Here, we find the energy of the tank–ship system. The effects of the ocean are added later.

As generalized position, we choose

$$q \triangleq [x^{n\top}, \eta^\top, q_t]^\top \in \mathbb{R}^8, \tag{30}$$

where x^n is the position of the body frame relative to the inertial frame (in the inertial frame), η the unit quaternion representing the orientation of the body frame relative to the inertial frame and q_t is the tank state defined in (21).

We note that, due to using unit quaternions, the dimension of the generalized position vector is greater than the number of degrees of freedom (eight versus seven), so we will have to use Hamiltonian rather than Lagrangian mechanics (Goldstein et al., 2002; Lanczos, 1970). Using Euler angles would obviate the need for Hamiltonian mechanics, but comes with the drawbacks mentioned in Section 3.

The time derivative of q can be found from (3) and Properties 6. We define $\mathcal{P}$ as

$$\mathcal{P}(\eta) \triangleq \begin{bmatrix} R^\top(\eta) & 0_{3\times4} & 0_{3\times1} \\ 0_{4\times3} & \frac{1}{2}G(\eta) & 0_{4\times1} \\ 0_{1\times3} & 0_{1\times4} & 1 \end{bmatrix} \in \mathbb{R}^{8\times7} \tag{31}$$

and the generalized velocity vector ν as

$$\nu \triangleq \left[v^{b\top}, \omega^{b\top}, \dot{q}_t\right]^\top \in \mathbb{R}^7 \tag{32}$$

such that

$$\dot{q} = \mathcal{P}^\top(\eta)\nu. \tag{33}$$

We also define the matrix

$$P(\eta) \triangleq \begin{bmatrix} R^\top(\eta) & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 2G(\eta) & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times4} & 1 \end{bmatrix}, \tag{34}$$

and note that

$$\nu = P(\eta)\dot{q}.$$

Proposition 1. *The potential energy U of the ship–tank system is given by*

$$U(q) = gm_0h_0 - ge_z^\top R\left[mr_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)r_t^b(\sigma)\,\mathrm{d}\sigma\right] \tag{35}$$

where g is the acceleration of gravity, m_0 is the total mass of the ship and the tank fluid, h_0 is an arbitrary constant, $e_z = [0,0,1]^\top$, m is the mass of the ship (excluding the mass of the tank fluid) and ρ_t is the density of the tank fluid.

Proof. See Appendix A. □

Proposition 2. *The complimentary kinetic energy of the system $\tilde{T}$[3] is given by*

$$\tilde{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top \mathcal{M}(q)\dot{q} \tag{36}$$

where

$$\mathcal{M}(q) = P^\top(\eta)M_t(q_t)P(\eta) = \mathcal{M}^\top \in \mathbb{R}^{8\times8} \tag{37}$$

and $M_t(q_t) \in \mathbb{R}^{7\times7}$ is the inertia matrix of the ship–tank system, defined in (B.4).

Proof. See Appendix B. □

[3]The complimentary kinetic energy is the kinetic energy as a function of $\dot{q}$ rather than the generalized velocity ν, but taking the same value, i.e., $\tilde{T}(q,\dot{q}) = T(q,\nu)$ if T is the kinetic energy.

Note that $\det(\mathcal{M}) = 0$; $\mathcal{M}$ is singular since the system is over-parametrized (there are seven degrees of freedom, but $\dim q = 8$).

As a complement to $\mathcal{M}$, we define the matrix $\mathcal{W}$ as

$$\mathcal{W}(q) \triangleq \mathcal{P}^\top(\eta) M_t^{-1}(q_t) \mathcal{P}(\eta) = \mathcal{W}^\top(q). \tag{38}$$

The matrices P, $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{W}$ satisfy several interesting properties. These are listed in Appendix C.

5.1. Hamilton's equations for the ship–tank system

The derivation of Hamilton's equations for the ship–tank system are fairly long and technical. For increased readability, this section provides a summary, while the full details can be found in Appendix E.

Unlike Lagrangian dynamics, which uses the generalized position q and its derivative $\dot{q}$ as variables, Hamiltonian mechanics uses the generalized position and generalized momentum. These have the same dimensions. By definition, the generalized momentum is found from the complimentary kinetic energy. Letting $\tilde{p} \in \mathbb{R}^8$ be the generalized momentum,

$$\tilde{p} \triangleq \frac{\partial \tilde{T}}{\partial \dot{q}} = \mathcal{M}(q)\dot{q} \in \mathbb{R}^8. \tag{39}$$

However, we know that $\tilde{p}$ has higher dimension than is, strictly speaking, necessary. There are only seven degrees of freedom, while $\dim \tilde{p} = \dim q = 8$. We can therefore get a somewhat more useful momentum in $\mathbb{R}^7$ by defining

$$p \triangleq M_t(q_t)\nu \in \mathbb{R}^7 \tag{40}$$

and note that (dropping function arguments for brevity) $\tilde{p} = P^\top M_t P \dot{q} = P^\top M_t \nu = P^\top p$, which implies $p = \mathcal{P}\tilde{p}$ by Property 13 (see Appendix C).

Proposition 3. *The dynamics of the ship–tank system are given by*

$$\dot{q} = \mathcal{P}^\top(q) M_t^{-1}(q) p \tag{41}$$

$$\dot{p} = \tau(t) - C_t(q,p) M_t^{-1}(q) p - k_t(q). \tag{42}$$

where M_t is the inertia matrix of the ship–tank system defined in (B.4), *$\tau \in \mathbb{R}^7$ are generalized forces related to those of the virtual work principle in Appendix D, $C_t = -C_t^\top \in \mathbb{R}^{7\times 7}$ is a Coriolis/centripetal force matrix defined in* (E.42) *and*

$$k_t(q) \triangleq g \begin{bmatrix} 0_{3\times 1} \\ S(R^\top e_z)\left[m r_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma) A(\sigma)\,\mathrm{d}\sigma\right] \\ -\rho_t A_0 e_z^\top R\left[r_t^b(\varsigma_p) - r_t^b(-\varsigma_s)\right] \end{bmatrix} \in \mathbb{R}^7 \tag{43}$$

is a vector of restoring forces.

Proof. See Appendix E. □

6. Total dynamics for the ship–tank–ocean system

There are other, external, forces acting on the ship–tank system that were not included in the dynamics of Proposition 3 (other than through τ). There are potential and viscous damping effects on the ship (Fossen, 2011), damping effects on the tank fluid (Goodrich, 1968; Holden et al., 2011; Kagawa et al., 1989; Lloyd, 1989, 1998; Marzouk and Nayfeh, 2009; Moaleji and Greig, 2006, 2007; Neves et al., 2009; Sellars and Martin, 1992), buoyancy on the ship (Fossen, 2011), environmental forces (Fossen, 2011), control forces and some gravity-induced forces. Hydrostatic buoyancy is actually a potential force, but buoyancy is rendered very complex in the presence of waves and was not included in the previous section.

To add these extra forces, we will need to make one final assumption:

Assumption 11. Generalized forces associated with the virtual work defined in Section Appendix D obey the superposition principle.

We can then write the external forces as

$$\tau = \tau_g + \tau_p + \tau_d + \tau_c + \tau_e \in \mathbb{R}^7 \tag{44}$$

where τ_g are gravity forces not already included, τ_p are pressure forces, τ_d are damping forces, τ_c are control forces and τ_e are unmodeled forces.

Buoyancy is included in the term τ_p, and is non-zero even in calm water for a ship at rest. The weight of the ship will counteract some or all of this (depending on whether or not the ship is floating). This weight was not included in the term k_t, as the potential energy of the ship and thus its weight does not depend on the ship's position, only on its orientation. We therefore add this force manually as (Fossen, 2011)

$$\tau_g(q) = m_0 g \begin{bmatrix} R^\top(\eta) e_z \\ 0_{4\times 1} \end{bmatrix}. \tag{45}$$

The pressure forces τ_p are caused by the pressure of the surrounding ocean. These do not affect the tank directly, since the tank fluid is not in contact with the ocean. We let $\tau_p = [\bar{\tau}_p^\top, 0]^\top$ where $\bar{\tau}_p \in \mathbb{R}^6$ is the pressure forces on the ship. From Fossen (2011), these forces are

$$\bar{\tau}_p = -\bar{M}_A \dot{\bar{\nu}} - D_p \bar{\nu} - \bar{C}_A(\bar{\nu})\bar{\nu} - k_p(\bar{q}, t) \tag{46}$$

where $\bar{q} = [x^{n\top}, \eta^\top]^\top \in \mathbb{R}^7$ is the generalized position and $\bar{\nu} \triangleq [v^{b\top}, \omega^{b\top}]^\top \in \mathbb{R}^6$ the generalized velocity of the ship, $\bar{M}_A \in \mathbb{R}^{7\times 7}$ is the added mass matrix, $\bar{D}_p \in \mathbb{R}^{7\times 7}$ the potential damping matrix, $\bar{C}_A \in \mathbb{R}^{7\times 7}$ is the matrix associated with the Coriolis/centripetal forces due to added mass, and $k_p \in \mathbb{R}^7$ is restoring forces due to hydrostatic and hydrodynamic effects. The exact nature of the terms in (46) depend on the sea state, and will not be further analyzed in this work.

The damping term $D_p\bar{\nu}$ only contains potential damping. There are also other dissipative effects, such as vortex shedding and viscous damping (Faltinsen, 1998). In addition, there is also the damping in the tank. We therefore write the remaining generalized damping force τ_d as

$$\tau_d = -\begin{bmatrix} D_v(\bar{\nu})\bar{\nu} \\ d_t(\dot{q}_t)\dot{q}_t \end{bmatrix}. \tag{47}$$

We gather all damping terms in the matrix D, such that

$$\begin{bmatrix} D_p\bar{\nu} + D_v(\bar{\nu})\bar{\nu} \\ d_t(\dot{q}_t)\dot{q}_t \end{bmatrix} = D(\nu)\nu \tag{48}$$

with

$$D(\nu) \triangleq \begin{bmatrix} D_p + D_v(\bar{\nu}) & 0_{6\times 1} \\ 0_{1\times 6} & d_t(\dot{q}_t) \end{bmatrix}. \tag{49}$$

We define the added mass matrix in $\mathbb{R}^{7\times 7}$, M_A as

$$M_A \triangleq \begin{bmatrix} \bar{M}_A & 0_{6\times 1} \\ 0_{1\times 6} & 0 \end{bmatrix},$$

the added mass-induced Coriolis/centripetal matrix in $\mathbb{R}^{7\times 7}$, C_A as

$$C_A(\nu) \triangleq \begin{bmatrix} \bar{C}_A(\nu) & 0_{6\times 1} \\ 0_{1\times 6} & 0 \end{bmatrix} = -C_A^\top(\nu),$$

the total inertia matrix M as

$$M(q_t) \triangleq M_t(q_t) + M_A \tag{50}$$

and the total generalized restoring force k as

$$k(q,t) \triangleq k_t(q) - \tau_g(q) + \begin{bmatrix} k_p(\bar{q},t) \\ 0 \end{bmatrix} = \begin{bmatrix} k_p(\bar{q},t) + k_g(q) \\ -\rho_t A_0 e_z^\top R(\eta)\left[r_t^b(\varsigma_p(q_t)) - r_t^b(-\varsigma_s(q_t))\right] \end{bmatrix} \tag{51}$$

with

$$k_g(q) \triangleq g \begin{bmatrix} m_0 R^\top(\eta) e_z \\ S(R^\top(\eta)e_z)\left[m r_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right] \end{bmatrix} \in \mathbb{R}^6. \tag{52}$$

The resulting dynamics is then given by

$$\dot{q} = \mathcal{P}^\top M_t^{-1} p \tag{53}$$

$$\dot{p} = \tau_e(t) + \tau_c(t) - M_A\dot{\nu} - \left[C_t(q_t,p) + D(\nu) + C_A(\nu)\right] M_t^{-1} p - k(q,t). \tag{54}$$

The presence of the added mass term (which gives a generalized force proportional to generalized acceleration) makes it cumbersome to use generalized momentum as a state variable.

We find

$$\begin{aligned} \dot{p} &= M_t\dot{\nu} + \dot{M}_t\nu = M_t\dot{\nu} + \dot{q}_t\frac{\partial M_t}{\partial q_t}\nu \\ &= \tau_e(t) + \tau_c(t) - M_A\dot{\nu} - \left[C_t(q_t,p) + D(\nu) + C_A(\nu)\right] M_t^{-1} p - k(q,t) \\ &= \tau_e(t) + \tau_c(t) - M_A\dot{\nu} - \left[C_t(q_t,M_t\nu) + D(\nu) + C_A(\nu)\right] \nu - k(q,t). \end{aligned} \tag{55}$$

We rearrange the terms in the above equality and get

$$M(q_t)\dot{\nu} = \tau_e(t) + \tau_c(t) - \left[C(q_t,\nu) + D(\nu)\right]\nu - k(q,t) \tag{56}$$

where

$$C(q_t,\nu) \triangleq \dot{q}_t \frac{\partial M_t}{\partial q_t} + C_t(q_t, M_t(q_t)\nu) + C_A(\nu)$$

is the total Coriolis/centripetal matrix.

Dropping function arguments for brevity, we can rewrite $C(q_t,\nu)$ as

$$\begin{aligned}
C &= \left[\begin{array}{cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$C_s + \bar{C}_A$}} & -\frac{\partial M_\nu}{\partial q_t}\omega^b \\
 & & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \\
-\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} & -\frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0
\end{array}\right] \\
&\quad - \dot{q}_t \begin{bmatrix}
0_{3\times3} & -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times1} \\
0_{3\times3} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & 0_{3\times1} \\
\frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t}
\end{bmatrix} + \dot{q}_t \begin{bmatrix}
0_{3\times3} & -\frac{\partial M_\nu}{\partial q_t} & \frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\
\frac{\partial M_\nu}{\partial q_t} & \frac{\partial M_\omega}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\
\frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{77}}{\partial q_t}
\end{bmatrix} \\
&= \left[\begin{array}{cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$C_s + \bar{C}_A$}} & -\frac{\partial M_\nu}{\partial q_t}\omega^b \\
 & & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \\
-\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} & -\frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0
\end{array}\right] + \dot{q}_t \begin{bmatrix}
0_{3\times3} & 0_{3\times3} & \frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\
\frac{\partial M_\nu}{\partial q_t} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\
0_{1\times3} & 0_{1\times3} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t}
\end{bmatrix} \\
&= C_w + \dot{q}_t C_m
\end{aligned} \tag{57}$$

with

$$\begin{aligned}
C_w &\triangleq \left[\begin{array}{cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$C_s + \bar{C}_A$}} & -\frac{\partial M_\nu}{\partial q_t}\omega^b \\
 & & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \\
-\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} & -\frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0
\end{array}\right] = -C_w^\top \\
C_m &= \begin{bmatrix}
0_{3\times3} & 0_{3\times3} & \frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\
\frac{\partial M_\nu}{\partial q_t} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\
0_{1\times3} & 0_{1\times3} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t}
\end{bmatrix}.
\end{aligned}$$

We note that M and C satisfy the following property:

Property 10. *The matrix $\dot{M} - 2C$ is skew-symmetric, that is*

$$\dot{M} - 2C = -\left(\dot{M} - 2C\right)^\top \Leftrightarrow w^\top\left(\dot{M} - 2C\right)w = 0 \ \forall\, w \in \mathbb{R}^7. \tag{58}$$

Proof. We already know that C_w is skew-symmetric, so the statement (58) is equivalent to the statement $\dot{M} - 2\dot{q}_t C_m = -\left(\dot{M} - 2\dot{q}_t C_m\right)^\top$. Since M is only a function of q_t, we have

$$
\begin{aligned}
\dot{M} - 2\dot{q}_t C_m &= \dot{q}_t \frac{\partial M}{\partial q_t} - 2\dot{q}_t C_m = \dot{q}_t \left(\frac{\partial M_t}{\partial q_t} - 2C_m \right) \\
&= \dot{q}_t \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & \frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\ \frac{\partial M_\nu}{\partial q_t} & \frac{\partial M_\omega}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{77}}{\partial q_t} \end{bmatrix} - \dot{q}_t \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & 2\frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\ 2\frac{\partial M_\nu}{\partial q_t} & \frac{\partial M_\omega}{\partial q_t} & 2\frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\ 0_{1\times 3} & 0_{1\times 3} & \frac{\partial m_{77}}{\partial q_t} \end{bmatrix} \\
&= \dot{q}_t \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & -\frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\ -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times 3} & -\frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & 0 \end{bmatrix} = -\left(\dot{M} - 2\dot{q}_t C_m\right)^\top
\end{aligned}
$$

since $M_\nu = -M_\nu^\top$ implies $\partial M_\nu / \partial q_t = -\partial M_\nu^\top / \partial q_t$. Proof of the equivalency of the two statements in (58) can be found in Kreyszig (1999). □

We can then define the dynamics for the full 7-DOF ship–tank–ocean system (main result):

Definition 1 (7-DOF model for arbitrarily shaped u-tanks (unit quaternion version)). The dynamics of the full 7-DOF u-tank model with a generic u-tank, using unit quaternions to represent the orientation of the ship, can be written as

$$\dot{q} = \mathcal{P}^\top(\eta)\nu \tag{59}$$

$$M(q_t)\dot{\nu} = \tau_e(t) + \tau_c(t) - C(q_t, \nu)\nu - D(\nu)\nu - k(q, t) \tag{60}$$

where $\mathcal{P}$ is defined in (31), M in (50), C in (57), D in (49), k in (51), τ_e is the unmodeled disturbances and τ_c the control forces.

It is possible to derive an Euler angle version of the model in Definition 1 from (59)–(60). For the convenience of those who prefer Euler angles, an Euler angle version of the model in Definition 1 is here presented.

Definition 2 (7-DOF model for arbitrarily shaped u-tanks (Euler angle version)). The dynamics of the full 7-DOF u-tank model with a generic u-tank, using Euler angles to represent the orientation of the ship, can be written as

$$\dot{q}_e = \mathcal{P}_e(\Theta)\nu \tag{61}$$

$$M(q_t)\dot{\nu} = \tau_e(t) + \tau_c(t) - C(q_t, \nu)\nu - D(\nu)\nu - k_e(q_e, t) \tag{62}$$

where $\Theta = [\phi, \theta, \psi]^\top \in \mathbb{R}^3$ is a vector of roll–pitch–yaw Euler angles; $q_e = [x^{n\top}, \Theta^\top, q_t]^\top \in \mathbb{R}^7$ is generalized position;

$$\mathcal{P}_e = \begin{bmatrix} R_e(\Theta) & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & G_e(\Theta) & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{7\times 7}$$

with

$$R_e(\Theta) = \begin{bmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$G_e(\Theta) = \begin{bmatrix} 1 & \mathrm{s}_\phi \mathrm{t}_\theta & \mathrm{c}_\phi \mathrm{t}_\theta \\ 0 & \mathrm{c}_\phi & -\mathrm{s}_\phi \\ 0 & \mathrm{s}_\phi/\mathrm{c}_\theta & \mathrm{c}_\phi/\mathrm{c}_\theta \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathrm{s.} = \sin(\cdot), \mathrm{c.} = \cos(\cdot), \mathrm{t.} = \tan(\cdot)$;

$$k_e(q_e, t) = \begin{bmatrix} k_{p,e}(q_e, t) + k_{g,e}(q_e) \\ -\rho_t A_0 e_z^\top R_e(\Theta) \left[r_t^b(\varsigma_p(q_t)) - r_t^b(-\varsigma_s(q_t)) \right] \end{bmatrix}$$

with $k_{p,e}$ being generalized pressure forces having the same numerical value as k_p (i.e., $k_{p,e}(q_e, t) = k_p(\bar{q}, t)$) and

$$k_{g,e}(q_e) = g \begin{bmatrix} m_0 R_e^\top(\Theta) e_z \\ S(R_e^\top(\Theta) e_z) \left[m r_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma) A(\sigma)\, \mathrm{d}\sigma \right] \end{bmatrix} \in \mathbb{R}^6;$$

and the other matrices are as in Definition 1.

7. Comparison to existing models

Most existing models of u-tanks (e.g., Holden et al. (2011); Kagawa et al. (1989); Lloyd (1989, 1998); Moaleji and Greig (2007); Sellars and Martin (1992)) are for u-tanks that consist of three rectangular prisms; one for each reservoir and one for the duct (see Figure 3).

A rectangular-prism u-tank can be described by

$$y_t^b(\sigma) = \begin{cases} \frac{w}{2} & \forall \quad \sigma \in (-\infty, -w/2] \\ -\sigma & \forall \quad \sigma \in [-w/2, w/2] \\ -\frac{w}{2} & \forall \quad \sigma \in [w/2, \infty) \end{cases} \tag{63}$$

$$z_t^b(\sigma) = \begin{cases} r_d + \frac{w}{2} + \sigma & \forall \quad \sigma \in (-\infty, -w/2] \\ r_d & \forall \quad \sigma \in [-w/2, w/2] \\ r_d + \frac{w}{2} - \sigma & \forall \quad \sigma \in [w/2, \infty) \end{cases} \tag{64}$$

$$A(\sigma) = \begin{cases} A_r & \forall \quad \sigma \in (-\infty, -w/2] \\ A_d & \forall \quad \sigma \in [-w/2, w/2] \\ A_r & \forall \quad \sigma \in [w/2, \infty) \end{cases} \tag{65}$$

Since the tank has rectangular cross-section, we define

$$A_r \triangleq w_r l_r\,, \quad A_d \triangleq h_d l_d. \tag{66}$$

The parameters in (63)–(66) have the interpretation indicated in Figure 3.[4]

Furthermore, $\varsigma_0 = h_t + \frac{w}{2}$. The value of A_0 is arbitrary. We choose $A_0 = A_r$.

The functions for the u-tank is sufficiently simple such that it is possible to explicitly solve (21):

$$q_t = \frac{1}{A_0} \int_{\varsigma_0}^{\varsigma_p} A(\sigma)\, \mathrm{d}\sigma = \varsigma_p - h_t - \frac{w}{2} \tag{67}$$

$$q_t = \frac{-1}{A_0} \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\, \mathrm{d}\sigma = h_t + \frac{w}{2} - \varsigma_s \tag{68}$$

as long as the duct is always full of fluid (which we will assume).

7.1. Comparison to the model of Lloyd (1989, 1998)

Like many other u-tank models (Faltinsen and Timokha, 2009; Moaleji and Greig, 2007; Sellars and Martin, 1992), the model of Lloyd (1989, 1998) has four degrees of freedom; namely sway, roll, yaw and a tank state. The model of Lloyd (1989, 1998) is also linear (i.e., only valid for low-amplitude motion), and only valid for a rectangular-prism tank.

To be able to compare the new model to that of Lloyd (1989, 1998), we will first take the 7-DOF model of Definitions 1 and 2, reduce it to the same four degrees of freedom as Lloyd's model, use a rectangular-prism u-tank (as defined in (63)–(65)) and linearize the dynamics. We will then define Lloyd's model, which was made under similar assumptions (with the extra assumption that $l_r = l_d$).

To reduce the order of the 7-DOF model, we use the Euler angle model of Definition 2. To derive the reduced-order model, we assume that $x = z = \theta = v_1^b = v_3^b = \omega_2^b = 0$.

We define

$$q_{r_4} \triangleq [y, \phi, \psi, q_t]^\top \tag{69}$$

$$\nu_{r_4} \triangleq [v_2^b, \omega_1^b, \omega_3^b, \dot{q}_t]^\top \tag{70}$$

and find that

$$\dot{q}_{r_4} = \begin{bmatrix} \cos(\phi)\cos(\psi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \nu_{r_4} \approx \nu_{r_4}$$

for small angles ϕ, ψ.

We also note that, using roll–pitch–yaw Euler angles, $R_e^\top e_z = [0, \sin(\phi), \cos(\phi)]^\top$ since θ is assumed zero.

[4]Technically, a rectangular-prism u-tank does not fit into the framework developed in this work; the tank centerline function is not a C^1 function. However, (63)–(65) can be approximated by a C^1 to arbitrary precision by interpolating (see Holden (2011, (8.1)–(8.3) p. 124)). Formally, we should use this approximation. In this paper, we will simply use (63)–(65) directly.

We define

$$M_{r_4} \triangleq \begin{bmatrix} m_{22}(0)+m_{A,22} & m_{24}(0)+m_{A,24} & m_{26}(0)+m_{A,26} & m_{27}(0) \\ m_{24}(0)+m_{A,42} & m_{44}(0)+m_{A,44} & m_{46}(0)+m_{A,46} & m_{47}(0) \\ m_{26}(0)+m_{A,62} & m_{46}(0)+m_{A,64} & m_{66}(0)+m_{A,66} & m_{67}(0) \\ m_{27}(0) & m_{47}(0) & m_{67}(0) & m_{77}(0) \end{bmatrix} \tag{71}$$

where m_{ij} is the i,jth element of M_t and $m_{A,ij}$ is the i,jth element of $\bar{M}_A$ (see (46)). We calculate

$$\begin{aligned}
m_{22}(0) &= m_{22}(q_t) = m_0 = m + \rho_t\left(2h_t w_r l_r + w h_d l_d\right) \\
m_{24}(0) &= \rho_t \int_{-\varsigma_0}^{\varsigma_0} A z_t^b \, \mathrm{d}\sigma - m z_g = -\rho_t\left(2x_r l_r r_d h_t + h_d l_d r_d w - w_r l_r h_t^2\right) - m z_g \\
m_{26}(0) &= -\rho_t \int_{-\varsigma_0}^{\varsigma_0} A z_t^b \, \mathrm{d}\sigma + m x_g = \rho_t x_t^b\left(2h_t w_r l_r + w h_d l_d\right) + m x_g \\
m_{27}(0) &= \rho_t A_0 \int_{-\varsigma_0}^{\varsigma_0} \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma} \, \mathrm{d}\sigma = -\rho_t w_r l_r w \\
m_{44}(0) &= I_{11} + \rho_t \int_{-\varsigma_0}^{\varsigma_0} [(y_t^b)^2 + (z_t^b)^2] A \, \mathrm{d}\sigma \\
&= I_{11} + \rho_t h_d l_d w\left(r_d^2 - \frac{w^2}{12}\right) + \rho_t\left(2r_d^2 + \frac{w^2}{2} + \frac{2}{3}h_t^2 - 2r_d h_t\right) w_r l_r h_t \\
m_{46}(0) &= I_{13} - \rho_t \int_{-\varsigma_0}^{\varsigma_0} A x_t^b z_t^b \, \mathrm{d}\sigma \\
&= I_{13} - 2\rho_t w_r l_r x_t^b r_d h_t - \rho_t h_d l_d r_d x_t^b w + \rho_t w_r l_r x_t^b h_t^2 \\
m_{47}(0) &= \rho_t A_0 \int_{-\varsigma_0}^{\varsigma_0} \left(y_t^b \frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma} - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma} z_t^b\right) \mathrm{d}\sigma = \rho_t w_r l_r w (h_t + r_d) \\
m_{66}(0) &= I_{33} + \rho_t \int_{-\varsigma_0}^{\varsigma_0} A\left[(x_t^b)^2 + (y_t^b)^2\right] \mathrm{d}\sigma \\
&= I_{33} + \left(2(x_t^b)^2 + \frac{1}{2}w^2\right)\rho_t w_r l_r h_t + \left((x_t^b)^2 + \frac{1}{12}w^2\right)\rho_t h_d l_d w \\
m_{67}(0) &= \rho_t A_0 \int_{-\varsigma_0}^{\varsigma_0} \left(x_t^b \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma} - \frac{\mathrm{d}\bar{x}_t^b}{\mathrm{d}\sigma} y_t^b\right) \mathrm{d}\sigma = \rho_t w_r l_r w x_t^b \\
m_{77}(0) &= \rho_t \int_{-\varsigma_0}^{\varsigma_0} \frac{A_0^2}{A} \, \mathrm{d}\sigma = \rho_t\left(2h_t w_r l_r + \frac{w_r^2 l_r^2 w}{h_d l_d}\right)
\end{aligned}$$

where I_{ij} is the i,jth element of the ship's moment of inertia matrix I.

Furthermore,

$$D_{r_4} = \begin{bmatrix} d_{22}(0) & d_{24}(0) & d_{26}(0) & 0 \\ d_{42}(0) & d_{44}(0) & d_{46}(0) & 0 \\ d_{62}(0) & d_{64}(0) & d_{66}(0) & 0 \\ 0 & 0 & 0 & d_t(0) \end{bmatrix} \tag{72}$$

where d_{ij} is the i,jth element of D of Definition 1 is the linear damping matrix.

Finally,

$$\tau_{e,r_4}(t) = B_{r_4}\tau_e(t) \tag{73}$$

$$\tau_{c,r_4}(t) = B_{r_4}\tau_c(t) \tag{74}$$

with

$$B_{r_4} \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{75}$$

are the unmodeled and control forces, respectively.

The restoring force matrix K_{r_4} can be found from the reduced-order spring term k_{r_4} (the nonlinear restoring force for the 4-DOF model). If $k_{p,i}$ is the ith element of $k_{p,e}$ of Definition 1, then – dropping function arguments for brevity – k_{r_4} is given by

$$\begin{aligned} k_{r_4} &= \begin{bmatrix} k_{p,2} + [0,1,0] m_0 g R^\top e_z \\ k_{p,4} + g[1,0,0] S(R^\top e_z) \left[m r_g^b + \rho_t \int_{-\varsigma_s}^{\varsigma_p} r_t^b A \, \mathrm{d}\sigma \right] \\ k_{p,6} + g[0,0,1] S(R^\top e_z) \left[m r_g^b + \rho_t \int_{-\varsigma_s}^{\varsigma_p} r_t^b A \, \mathrm{d}\sigma \right] \\ -g\rho_t A_0 e_z^\top R \left[r_t^b(\varsigma_p) - r_t^b(-\varsigma_s) \right] \end{bmatrix} \\ &= \begin{bmatrix} k_{p,2} + m_0 g \sin(\phi) \\ \Big(k_{p,4} + g m z_g \sin(\phi) + g\rho_t w w_r l_r q_t \cos(\phi) \\ \quad + g\rho_t \left[w_r l_r (2 r_d h_t - h_t^2 - q_t^2) + h_d l_d r_d w \right] \sin(\phi) \Big) \\ k_{p,6} - g \left[x_t^b m_t + m x_g \right] \sin(\phi) \\ g\rho_t w_r l_r w \sin(\phi) + 2 g\rho_t w_r l_r q_t \cos(\phi) \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} K_{r_4} &= \left. \frac{\partial k_{r_4}}{\partial q_{r_4}} \right|_{q_e = 0} \\ &= \begin{bmatrix} k_{p,22} & k_{p,24} + m_0 g & k_{p,26} & 0 \\ k_{p,42} & k_{p,44} + g m z_g + g\rho_t \left[w_r l_r h_t (2 r_d - h_t) + h_d l_d r_d w \right] & k_{p,46} & g\rho_t w_r l_r w \\ k_{p,62} & k_{p,64} - g \left[x_t^b m_t + m x_g \right] & k_{p,66} & 0 \\ 0 & g\rho_t w_r l_r w & 0 & 2 g\rho_t w_r l_r \end{bmatrix} \end{aligned} \tag{76}$$

where

$$k_{p,ij} = \left. \frac{\partial k_{p,i}}{\partial q_{e,j}} \right|_{q_e = 0}$$

with $q_{e,j}$ is the jth element of q_e are the partial derivatives of the pressure restoring forces.[5] The pressure force parameters may be time-varying (if the pressure field itself is time-varying, such as due to waves), and some may be zero. This renders K_{r_4} a function of time.

Definition 3 (Linearized 4-DOF rectangular-prism u-tank model). The linearized 4-DOF dynamics for a rectangular-prism u-tank are given by

$$M_{r_4} \ddot{q}_{r_4} + D_{r_4} \dot{q}_{r_4} + K_{r_4}(t) q_{r_4} = \tau_{e,r_4}(t) + \tau_{c,r_4}(t) \tag{77}$$

[5]It may be worth noting that the terms $g\rho_t \left[w_r l_r h_t (2 r_d - h_t) + h_d l_d r_d w \right] \phi$ and $-g x_t^b m_t \psi$ are the linearized moments due the the mass of the tank fluid in, respectively, roll and yaw.

where M_{r_4} is defined in (71), D_{r_4} in (72), K_{r_4} in (76), τ_{e,r_4} in (73) and τ_{c,r_4} in (74).

The model found in Lloyd (1989, 1998) is a linear, 4-DOF model, consisting of sway, roll, yaw and a tank state. The model is derived based on Newtonian mechanics. The model assumes a rectangular u-tank as in (63)–(65) with $l_r = l_d$.

We rewrite the model of Lloyd (1989, 1998) so that it has the same states as the model presented here.[6]

Definition 4 (The model of Lloyd). According to Lloyd (1989, 1998), the dynamics of a rectangular-prism u-tank is given by

$$M_L \ddot{q}_{r_4} + D_{r_4} \dot{q}_{r_4} + K_L q_{r_4} = \tau_{e,r_4}(t) + \tau_{c,r_4}(t) \tag{78}$$

where q_{r_4}, D_{r_4}, τ_{e,r_4} and τ_{c,r_4} are as in Definition 3, and

$$M_L = \begin{bmatrix} m + m_{A,22} & m_{A,24} & m_{A,26} & m_{27}(0) \\ m_{A,42} & I_{11} + m_{A,44} & m_{A,46} & m_{47}(0) \\ m_{A,62} & m_{A,64} & I_{33} + m_{A,66} & m_{67}(0) \\ m_{27}(0) & m_{47}(0) & m_{67}(0) & m_{77}(0) \end{bmatrix}$$

$$K_L = \begin{bmatrix} 0 & 0 & k_{p,26} & 0 \\ 0 & k_{p,44} + gmz_g & k_{p,46} & g\rho_t l_r w_r w \\ 0 & 0 & k_{p,66} & 0 \\ 0 & g\rho_t l_r w_r w & 0 & 2g\rho_t l_r w_r \end{bmatrix}$$

where matrix parameters are the same as in Definition 3.

This model is largely identical to the linear 4-DOF u-tank model of Definition 3. The main difference is in the mass matrix. In the model of Lloyd, the only effect on the inertia matrix of the ship–tank system is adding the coupling terms (m_{27}, m_{47} and m_{67}) and m_{77}. In the calculations shown in this work, the mass of the tank fluid will also change the other elements of the inertia matrix. Loading the tank with water will increase the ship's mass, and will also change its distribution. This will change the total mass of the ship, the center of gravity and the moment of inertia. This is not included in the model of Lloyd. However, the mass of the tank fluid is likely to be quite low compared to the mass of the ship (1–5% (Lloyd, 1989, 1998)), so this might not greatly affect the behavior of the model. The model of Lloyd also assumes that the body center of origin is the center of gravity, and that $I_{13} = 0$.

[6]Note that in Lloyd (1989, 1998), the states called are x_2, x_4, x_6 and τ. These equal, respectively, y, ϕ, ψ and q_t/w. Also note that there are some obvious typos in the original work Lloyd (1989, 1998). These are

Lloyd (1998, Equation (12.53a) p. 265) reads $a_{\tau 2} = -Q_t$, but should read $a_{\tau 2} = Q_t$.

Lloyd (1998, Equation (12.53d) p. 265) reads $a_{\tau 6} = -Q_t x_{B_1}$, but should read $a_{\tau 6} = Q_t x_{B_1}$.

Lloyd (1998, Equation (12.54b) p. 266)) reads $a_{42}\ddot{x}_2 + b_{42}\dot{x}_2 + (I_{44} + a_{44})\ddot{x}_4 + b_{44}\dot{x}_4 + c_{44}x_4 + a_{46}\ddot{x}_6 + b_{46}\dot{x}_6 + c_{46}x_6 - [a_{4\tau}\ddot{\tau} + c_{4\tau}\tau] = F_{w40}\sin(\omega_e t + \gamma_4)$, but should read $a_{42}\ddot{x}_2 + b_{42}\dot{x}_2 + (I_{44} + a_{44})\ddot{x}_4 + b_{44}\dot{x}_4 + c_{44}x_4 + a_{46}\ddot{x}_6 + b_{46}\dot{x}_6 + c_{46}x_6 + a_{4\tau}\ddot{\tau} + c_{4\tau}\tau = F_{w40}\sin(\omega_e t + \gamma_4)$.

These typos are independent of the reference frame used, and are propagated throughout Lloyd (1989, 1998).

In addition, there are a few differences in the spring term. However, the only differences related to ship–tank interaction is the lack of the terms $gx_t^b m_t$ and $g\rho_t\left[w_r l_r h_t(2r_d - h_t) + h_d l_d r_d w\right]$, which have been neglected in the model of Lloyd. These terms are related to shifting the center of gravity of the ship–tank system due to the mass of the tank fluid. Again, this change is likely to be small.[7]

Of course, the main advantage of the 7-DOF u-tank model of Definition 1 over the model of Lloyd is that while that model assumes one very specific tank shape, is completely linear and only 4-DOF, the new model can handle arbitrary u-tank shapes, is as nonlinear as is required, and has all seven degrees of freedom.

7.2. Comparison to the model of Holden et al. (2011)

Holden et al. (2011) presents several 2-DOF nonlinear models for a rectangular-prism u-tank with $l_r = l_d$ on a barge in calm water. These models were verified through small-scale experiments. We shall here compare the novel model of Definitions 1 and 2 to the model referred to as $\mathcal{L}$ in Holden et al. (2011)[8].

We also need to rewrite the model of Definitions 1 and 2 to a 2-DOF model. Defining $q_{r_2} \triangleq [\phi, q_t]^\top$ and using the same procedure as in Section 7.1 (except for the linearization), we can define a reduced-order model.

Definition 5 (2-DOF rectangular-prism u-tank model). The 2-DOF dynamics for a rectangular-prism u-tank are given by

$$M_{r_2}(q_t)\ddot{q}_{r_2} + C_{r_2}(q_t, \dot{q}_{r_2})\dot{q}_{r_2} + D_{r_2}(\dot{q}_{r_2})\dot{q}_{r_2} + k_{r_2}(q_{r_2}, t) = \tau_{e,r_2}(t) + \tau_{c,r_2}(t) \tag{79}$$

where

$$M_{r_2}(q_t) = \begin{bmatrix} m_{44}(q_t) + m_{A,44} & m_{47}(q_t) \\ m_{47}(q_t) & m_{77}(q_t) \end{bmatrix} \tag{80}$$

$$\begin{aligned} m_{44}(q_t) &= I_{11} + \rho_t \int_{-\varsigma_s}^{\varsigma_p} [(y_t^b)^2 + (z_t^b)^2] A \,\mathrm{d}\sigma \\ &= I_{11} + 2\rho_t w_r l_r (h_t - r_d) q_t^2 + \rho_t h_d l_d w \left(r_d^2 - \frac{w^2}{12} \right) + \rho_t \left(2r_d^2 + \frac{w^2}{2} + \frac{2}{3} h_t^2 - 2r_d h_t \right) w_r l_r h_t \end{aligned}$$

and calculations show that $m_{47}(q_t) = m_{47}(0), m_{77}(q_t) = m_{77}(0)$ and take the same value as the model of Definition 3,

$$C_{r_2}(q_t, \dot{q}_{r_2}) = 2\rho_t w_r l_r (h_t - r_d) q_t \dot{\phi} \begin{bmatrix} \dot{q}_t & \dot{\phi} \\ -\dot{\phi} & 0 \end{bmatrix} \tag{81}$$

$$D_{r_2}(\dot{q}_{r_2}) = \begin{bmatrix} \bar{d}_{44}(\dot{\phi}) & 0 \\ 0 & d_t(\dot{q}_t) \end{bmatrix} \tag{82}$$

[7]Lloyd's spring term is time-independent, since he assumes the pressure field is constant. This is in contrast to the new model, where the pressure field can be time-varying, such as due to waves.

[8]The other models are various simplifications and extensions of the model $\mathcal{L}$. Experiments performed in Holden et al. (2011) indicated that the extended models do not have significantly higher accuracy than the model $\mathcal{L}$, which is why the model of Definitions 1 and 2 is compared to the model $\mathcal{L}$ rather than one of its (more complicated) extensions.

with $\bar{d}_{44}(\dot{\phi}) = d_{44}(\bar{\nu})$ with all non-roll degrees of freedom set to 0,

$$k_{r_2}(q_{r_2}, t) = \begin{bmatrix} \bar{k}_4(q_{r_2}, t) \\ g\rho_t w_r l_r w \sin(\phi) + 2g\rho_t w_r l_r q_t \cos(\phi) \end{bmatrix} \tag{83}$$

with

$$\begin{aligned} \bar{k}_4(q_{r_2}, t) = \bar{k}_{p,4}(\phi, t) + gmz_g \sin(\phi) + g\rho_t w w_r l_r q_t \cos(\phi) \\ + g\rho_t \left[w_r l_r (2r_d h_t - h_t^2 - q_t^2) + h_d l_d r_d w \right] \sin(\phi) \end{aligned}$$

and $\bar{k}_{p,4}(\phi) = k_{p,4}(q_e)$ with all non-roll degrees of freedom set to 0, and

$$\tau_{e,r_2}(t) = B_{r_2} \tau_e(t) \tag{84}$$

$$\tau_{c,r_2}(t) = B_{r_2} \tau_c(t) \tag{85}$$

with

$$B_{r_2} \triangleq \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{86}$$

Definition 6 (The model $\mathcal{L}$). According to Holden et al. (2011), the dynamics of a rectangular-prism u-tank on a barge is given by

$$M_{r_2}(q_t)\ddot{q}_{r_2} + C_{r_2}(q_t, \dot{q}_{r_2})\dot{q}_{r_2} + D_{r_2}(\dot{q}_{r_2})\dot{q}_{r_2} + k_{\mathcal{L}}(q_{r_2}, t) = \tau_{e,r_2}(t) + \tau_{c,r_2}(t) \tag{87}$$

where M_{r_2}, C_{r_2}, D_{r_2}, τ_{e,r_2} and $\tau_{c,r_2}(t)$ are as in Definition 5 and

$$k_{\mathcal{L}}(q_{r_2}, t) = \begin{bmatrix} \tilde{k}_4(\phi) \\ g\rho_t w_r l_r w \sin(\phi) + 2g\rho_t w_r l_r q_t \cos(\phi) \end{bmatrix} \tag{88}$$

with

$$\begin{aligned} \tilde{k}_4(q_{r_2}, t) = \tilde{k}_{p,4}(\phi) + gmz_g \sin(\phi) + g\rho_t w w_r l_r q_t \cos(\phi) \\ + g\rho_t \left[w_r l_r (2r_d h_t - h_t^2 - q_t^2) + h_d l_d r_d w \right] \sin(\phi) \end{aligned}$$

where

$$\tilde{k}_{p,4}(\phi) = \frac{gm_0}{24D} \left[B^2 - 12D^2 + \frac{B^2}{\cos^2(\phi)} \right] \sin(\phi) \tag{89}$$

is the hydrostatic pressure moment in roll on a barge of dimensions $L \times B \times D$.

We note that this model is identical to that of Definition 5, except that the model $\mathcal{L}$ uses a specific function for the pressure forces rather than the undefined, generic function of the new model. This means that the verification results of Holden et al. (2011) are also valid for the new model, although only in reduced-order form for a specific u-tank.

8. Conclusions

This work presents a novel nonlinear 7-DOF model for ships equipped with a u-tank of arbitrary shape. The ship–tank interaction was modeled using Hamiltonian (analytical) mechanics, while external forces (such as those due to the surrounding ocean, actuators and various dissipative forces) were added later in a Newtonian framework. These external forces were not explicitly modeled in this work.

The new model is quite complex, and should achieve a high degree of accuracy. Its main strength is its ability to describe the motion of u-tanks of arbitrary shape. Furthermore, it can model high-amplitude motions due to its nonlinearity and it fully captures the coupling to all degrees of freedom. The main drawback is that the new model is quite complex, and contains certain integral relations that (except for very simple tank shapes) cannot be analytically solved. However, some of these complexities can easily be simplified away.

While experiments have not yet been performed to test the validity of the new model, it was compared to the experimentally verified (but significantly less powerful) model of Holden et al. (2011). Under similar assumptions, the two models are identical, and the verification results presented in Holden et al. (2011) should therefore still hold. The novel model was also compared to the commonly used (but also significantly less powerful) model of Lloyd (1989, 1998). The new model contains several effects left out of the model of Lloyd (1989, 1998), in addition to having the ability to describe a significantly broader class of u-tanks under much less strict conditions and with higher accuracy.

Appendix A. Potential energy of the ship–tank system

Any infinitesimal volume block $\mathrm{d}V$ of the tank or the ship at a position r^b has density $\rho(r^b)$ given by

$$\rho(r^b) = \begin{cases} \rho_t & \text{in the tank} \\ \rho_s(r^b) & \text{in the ship} \end{cases} \tag{A.1}$$

and is at a height $h(r^n)$ above some arbitrary zero point. We note that h is the zero level minus the inertial z-component of r^n, i.e.,

$$h(r^n) = h_0 - e_z^\top r^n = h_0 - e_z^\top R r^b \tag{A.2}$$

where $e_z^\top \triangleq [0, 0, 1]$ and h_0 is an arbitrary constant defining the zero point for potential energy. The negative signage is because the z-axis has the same direction as the gravity field.

The potential energy $\mathrm{d}U$ of $\mathrm{d}V$ is given by

$$\mathrm{d}U = g\rho(r^b)h(r^n)\mathrm{d}V = g\rho(r^b)\left(h_0 - e_z^\top R r^b\right)\mathrm{d}V. \tag{A.3}$$

The total potential energy U of the ship and the tank fluid is then given by

$$\begin{aligned} U &= \int_{\text{ship and tank}} \mathrm{d}U = g m_0 h_0 - g e_z^\top R \left[\int_{\text{ship}} \rho_s(r^b) r^b \,\mathrm{d}V + \rho_t \int_{\text{tank}} r_t^b \,\mathrm{d}V\right] \\ &= g m_0 h_0 - g e_z^\top R \left[m r_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) r_t^b(\sigma)\,\mathrm{d}\sigma\right] \end{aligned} \tag{A.4}$$

where

$$m_0 \triangleq \int_{\text{ship and tank}} \rho(r^b)\,\mathrm{d}V = \int_{\text{ship}} \rho_s(r^b)\,\mathrm{d}V + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)\,\mathrm{d}\sigma > 0$$

is the combined mass of the ship and the tank fluid, and

$$r_g^b \triangleq \frac{1}{m} \int_{\text{ship}} \rho_s(r^b) r^b\,\mathrm{d}V \tag{A.5}$$

is the definition of the (ship's) center of gravity.

We recognize (A.4) as matching Proposition 1.

Appendix B. Kinetic energy of the ship–tank system

An infinitesimal volume block $\mathrm{d}V$ of the tank or ship at a position r^b in the body frame has density $\rho(r^b)$ given by (A.1) and velocity $v_v^b(r^b)$ given by

$$v_v^b(r^b) = \begin{cases} v^b + \omega^b \times r^b + \frac{A_0 \dot{q}_t}{A(\sigma)} \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma) & \text{in the tank} \\ v^b + \omega^b \times r^b & \text{in the ship} \end{cases} \tag{B.1}$$

where v^b is the translational and ω^b the angular velocity of the ship. The velocity of the tank fluid comes from (29).

The volume block has kinetic energy $\mathrm{d}T$ given by

$$\mathrm{d}T = \frac{1}{2} \rho(r^b) \|v_v^b(r^b)\|_2^2 \mathrm{d}V. \tag{B.2}$$

The total kinetic energy T of the ship and the tank fluid is then given by

$$\begin{aligned}
T(q,\nu) &= \frac{1}{2} \int_{\text{tank and ship}} \rho(r^b) \|v_v^b(r^b)\|_2^2\,\mathrm{d}V \\
&= \frac{1}{2} \int_{\text{ship}} \rho_s(r^b) \left\| v^b + \omega^b \times r^b \right\|_2^2\,\mathrm{d}V \\
&\quad + \frac{1}{2}\rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) \left\| v^b + \omega^b \times r_t^b(\sigma) + \frac{A_0 \dot{q}_t}{A(\sigma)} \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma) \right\|_2^2 \mathrm{d}\sigma \\
&= \frac{1}{2} \left[\int_{\text{ship}} \rho_s(r^b)\,\mathrm{d}V + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)\,\mathrm{d}\sigma \right] \|v^b\|_2^2 \\
&\quad - \frac{1}{2} {\omega^b}^\top \left[\int_{\text{ship}} \rho_s(r^b) S^2(r^b)\,\mathrm{d}V + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) S^2(r_t^b(\sigma))\,\mathrm{d}\sigma \right] \omega^b \\
&\quad + {\omega^b}^\top \left[\int_{\text{ship}} \rho_s(r^b) S(r^b)\,\mathrm{d}V + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) S(r_t^b(\sigma))\,\mathrm{d}\sigma \right] v^b \\
&\quad + \rho_t A_0 \dot{q}_t {v^b}^\top \left[\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma)\,\mathrm{d}\sigma \right] + \frac{\rho_t A_0^2 \dot{q}_t^2}{2} \left[\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{1}{A(\sigma)}\,\mathrm{d}\sigma \right] \\
&\quad + \rho_t A_0 \dot{q}_t {\omega^b}^\top \left[\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} S(r_t^b(\sigma)) \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma)\,\mathrm{d}\sigma \right] \\
&= \frac{1}{2} \nu^\top M_t(q_t) \nu
\end{aligned} \tag{B.3}$$

where

$$M_t(q_t) \triangleq \begin{bmatrix} m_0\mathbb{I}_3 & -M_\nu(q_t) - mS(r_g^b) & m_{v,\dot{q}_t}(q_t) \\ M_\nu(q_t) + mS(r_g^b) & M_\omega(q_t) + I & m_{\omega,\dot{q}_t}(q_t) \\ m_{v,\dot{q}_t}^\top(q_t) & m_{\omega,\dot{q}_t}^\top(q_t) & m_{77}(q_t) \end{bmatrix} \tag{B.4}$$

is the inertia matrix and we have used that $\|r\|^2 = r^\top r$ for all vectors r. We note that $M_t = M_t^\top \in \mathbb{R}^{7\times 7}$ is a positive definite matrix.

The components of the inertia matrix are:

$$m_0 \triangleq \int_{\text{ship}} \rho_s(r^b)\,\mathrm{d}V + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)\,\mathrm{d}\sigma > 0 \tag{B.5}$$

is the total mass of the ship and tank fluid,

$$M_\nu(q_t) \triangleq \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) S(r_t^b(\sigma))\,\mathrm{d}\sigma = -M_\nu^\top(q_t) \in \mathbb{R}^{3\times 3} \tag{B.6}$$

$$mS(r_g^b) \triangleq \int_{\text{ship}} \rho_s(r^b) S(r^b)\,\mathrm{d}V = -mS^\top(r_g^b) \in \mathbb{R}^{3\times 3} \tag{B.7}$$

are the cross-couplings between the angular and translational accelerations,

$$M_\omega(q_t) \triangleq -\rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma) S^2(r_t^b(\sigma))\,\mathrm{d}\sigma = M_\omega^\top(q_t) > 0 \in \mathbb{R}^{3\times 3} \tag{B.8}$$

$$I \triangleq -\int_{\text{ship}} \rho_s(r^b) S^2(r^b)\,\mathrm{d}V = I^\top > 0 \in \mathbb{R}^{3\times 3} \tag{B.9}$$

are the moment of inertia tensors of the tank fluid and the ship,

$$m_{v,\dot{q}_t}(q_t) \triangleq \rho_t A_0 \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma)\,\mathrm{d}\sigma \in \mathbb{R}^{3\times 1} \tag{B.10}$$

$$m_{\omega,\dot{q}_t}(q_t) \triangleq \rho_t A_0 \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} S(r_t^b(\sigma)) \frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\sigma)\,\mathrm{d}\sigma \in \mathbb{R}^{3\times 1} \tag{B.11}$$

give the cross-coupling between the acceleration of the tank fluid and the rigid-body generalized velocities and

$$m_{77}(q_t) \triangleq \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{A_0^2}{A(\sigma)}\,\mathrm{d}\sigma > 0 \tag{B.12}$$

is the inertial mass of the tank fluid (which is distinct from the actual mass of the tank fluid).

As an intermediary step in Hamiltonian mechanics, the energy needs to be a function of the generalized coordinates q and their derivatives $\dot{q}$ (Goldstein et al., 2002; Lanczos, 1970). We therefore need to rewrite the kinetic energy as a function of $\dot{q}$.

Since $\nu = P(\eta)\dot{q}$, the complimentary kinetic energy $\tilde{T}$[9] is given by

$$\tilde{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top \mathcal{M}(q)\dot{q} \tag{B.13}$$

where

$$\mathcal{M}(q) \triangleq P^\top(\eta) M_t(q_t) P(\eta) = \mathcal{M}^\top(q) \in \mathbb{R}^{8\times 8}. \tag{B.14}$$

We recognize (B.13) as matching Proposition 2.

[9] The complimentary kinetic energy function $\tilde{T}$ is a function that takes the same value as the kinetic energy function T, but is a function of $\dot{q}$ rather than ν, i.e., $\tilde{T}(q,\dot{q}) = T(q,\nu)$.

Appendix C. Properties of the matrices $\boldsymbol{P}$, $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{W}}$

The matrices P, $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{W}$ satisfy several interesting properties. These are here summarized. In this appendix, function arguments are dropped for brevity.

Property 11. *$\mathcal{M}$ is symmetric and positive semidefinite.*

Proof. $\mathcal{M} = \mathcal{M}^\top \geq 0$ if it satisfies $w^\top \mathcal{M} w \geq 0 \ \forall \ w \in \mathbb{R}^7$. From the definition of $\mathcal{M}$, $w^\top \mathcal{M} w = w^\top P^\top M_t P w = \tilde{w}^\top M_t \tilde{w} \geq 0$ with $\tilde{w} = Pw$ since $M_t = M_t^\top > 0$. Thus $\mathcal{M}$ is at least positive semidefinite. It is not positive definite, however, because $w = [0_{1\times 3}, \eta^\top]^\top \neq 0 \Rightarrow w^\top \mathcal{M} w = 0$ since $G\eta = 0$ (by Property 4). $\square$

Property 12. *$\mathcal{W}$ is symmetric and positive semidefinite.*

Proof. $\mathcal{W} = \mathcal{W}^\top \geq 0$ if it satisfies $w^\top \mathcal{W} w \geq 0 \ \forall \ w \in \mathbb{R}^7$. From the definition of $\mathcal{W}$, $w^\top \mathcal{W} w = w^\top \mathcal{P}^\top M_t^{-1} \mathcal{P} w = \tilde{w}^\top M_t^{-1} \tilde{w} \geq 0$ with $\tilde{w} = \mathcal{P} w$ since $M_t = M_t^\top > 0$ implies $M_t^{-1} = M_t^{-\top} > 0$. Thus $\mathcal{W}$ is at least positive semidefinite. It is not positive definite, however, because $w = [0_{1\times 3}, \eta^\top]^\top \neq 0 \Rightarrow w^\top \mathcal{W} w = 0$ since $G\eta = 0$ (by Property 4). $\square$

Property 13. *$\mathcal{P}P^\top = \mathbb{I}_7$.*

Proof. We have

$$\mathcal{P}P^\top = \begin{bmatrix} R^\top R & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & GG^\top & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 1 \end{bmatrix} = \mathbb{I}_7$$

since $R^\top = R^{-1}$ and $GG^\top = \mathbb{I}_3$ by Property 3. $\square$

Property 14. *$P\mathcal{P}^\top = \mathbb{I}_7$.*

Proof. We have

$$P\mathcal{P}^\top = \begin{bmatrix} R^\top R & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & GG^\top & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 1 \end{bmatrix} = \mathbb{I}_7$$

since $R^\top = R^{-1}$ and $GG^\top = \mathbb{I}_3$ by Property 3. $\square$

Property 15. *$\mathcal{P}^\top P \mathcal{P}^\top = \mathcal{P}^\top$.*

Proof. We have $\mathcal{P}^\top P \mathcal{P}^\top = \mathcal{P}^\top \mathbb{I}_7 = \mathcal{P}^\top$ by Property 14. $\square$

Property 16. *$P^\top \mathcal{P} P^\top = P^\top$.*

Proof. We have $P^\top \mathcal{P} P^\top = P^\top \mathbb{I}_7 = P^\top$ by Property 13. $\square$

Property 17. *$\mathcal{W}\mathcal{M}\mathcal{W} = \mathcal{W}$.*

Proof. We have $\mathcal{W}\mathcal{M}\mathcal{W} = \mathcal{P}^\top M_t^{-1}\mathcal{P}P^\top M_t P\mathcal{P}^\top M_t^{-1}\mathcal{P} = \mathcal{P}^\top P\mathcal{P}^\top M_t^{-1}\mathcal{P} = \mathcal{P}^\top M_t^{-1}\mathcal{P} = \mathcal{W}$, where Properties 13 and 15 have been used. $\square$

Property 18. $\mathcal{M}\mathcal{W}\mathcal{M} = \mathcal{M}$.

Proof. We have $\mathcal{M}\mathcal{W}\mathcal{M} = P^\top M_t P\mathcal{P}^\top M_t^{-1}\mathcal{P}P^\top M_t P = P^\top \mathcal{P}P^\top M_t P = P^\top M_t P = \mathcal{M}$, where Properties 14 and 16 have been used. $\square$

Appendix D. Virtual work on the ship–tank system

Following Shivarama (2002); Shivarama and Fahrenthold (2004), we define the quasi-coordinates q_ω associated with the co-rotating components of the angular velocity so that $\dot{q}_\omega \triangleq \omega^b$.

The virtual work done by imposed forces $\tau_f^n(t) \in \mathbb{R}^3$ (on the ship) and $\tau_t(t) \in \mathbb{R}$ (on the tank fluid), and torques $\tau_m^b(t) \in \mathbb{R}^3$ (on the ship) is then given by

$$\delta W = \delta x^{n\top}\tau_f^n(t) + \delta q_\omega^\top \tau_m^b(t) + \delta q_t \tau_t(t). \tag{D.1}$$

Since $\delta q_\omega = 2G\delta\eta$ (by Property 6), we get

$$\delta W = \delta x^{n\top}\tau_f^n(t) + 2\delta\eta^\top G^\top \tau_m^b(t) + \delta q_t \tau_t(t) = \delta q^\top \begin{bmatrix} \tau_f^n \\ 2G^\top \tau_m^b \\ \tau_t \end{bmatrix}. \tag{D.2}$$

We therefore see that the vector of generalized forces associated with the virtual work δW is

$$\tilde{\tau} \triangleq \begin{bmatrix} \tau_f^n \\ 2G^\top \tau_m^b \\ \tau_t \end{bmatrix} \in \mathbb{R}^8. \tag{D.3}$$

Appendix E. Derivation of Hamilton's equations for the ship–tank system

Using the kinetic and potential energies of Section 5, all that is required to derive the dynamics of the system is a virtual work principle. This is defined in Appendix D.

Initially, the dynamics will be given using a generalized momentum in $\mathbb{R}^8$. This is somewhat inconvenient, as the generalized momentum can be expressed in $\mathbb{R}^7$ (as there are seven degrees of freedom). However, we need to use the more complex form as a stepping stone to get the more compact representation.

Appendix E.1. Generalized momentum and Hamiltonian

The generalized momentum $\tilde{p} \in \mathbb{R}^8$ can be found by

$$\tilde{p} = \frac{\partial \tilde{T}}{\partial \dot{q}} = \mathcal{M}\dot{q} \in \mathbb{R}^8. \tag{E.1}$$

Due to Properties 17 and 18, we can take

$$\dot{q} = \mathcal{W}\tilde{p}. \tag{E.2}$$

By the Legendre transform (see, e.g., Goldstein et al. (2002); Lanczos (1970)),

$$\begin{aligned} T = \tilde{p}^\top \dot{q} - \tilde{T} &= \tilde{p}^\top \mathcal{W}\tilde{p} - \frac{1}{2}\dot{q}^\top \mathcal{M}\dot{q} = \tilde{p}^\top \mathcal{W}\tilde{p} - \frac{1}{2}\tilde{p}^\top \mathcal{W}\mathcal{M}\mathcal{W}\tilde{p} \\ &= \tilde{p}^\top \mathcal{W}\tilde{p} - \frac{1}{2}\tilde{p}^\top \mathcal{W}\tilde{p} = \frac{1}{2}\tilde{p}^\top \mathcal{W}\tilde{p} \end{aligned} \tag{E.3}$$

where Properties 12 and 17 have been used.

Due to the shape of T and U, the Hamiltonian H is simply equal to the sum of the energy in the system (Goldstein et al., 2002), or

$$\begin{aligned} H(q,\tilde{p}) &= T(q,\tilde{p}) + U(q) \\ &= \frac{1}{2}\tilde{p}^\top \mathcal{W}\tilde{p} + gmz_g + g\rho_t \int_{-\varsigma_0}^{\varsigma_0} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma - ge_z^\top R\left[mr_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)r_t^b(\sigma)\,\mathrm{d}\sigma\right]. \end{aligned} \tag{E.4}$$

Appendix E.2. Using generalized momentum in $\mathbb{R}^8$

Since $\dim q = 8$, while there are only seven degrees of freedom, the system has a single algebraic constraint to satisfy:[10]

$$\xi(q,\tilde{p}) = \eta^\top \eta - 1 = 0. \tag{E.5}$$

We recognize this as representing the necessity that η remains a unit quaternion.

By Hamiltonian mechanics (Goldstein et al., 2002; Holden, 2011; Lanczos, 1970), the dynamics (with $q, \tilde{p} \in \mathbb{R}^8$) are given by

$$\dot{q} = \mathcal{W}\tilde{p} \tag{E.6}$$

$$\dot{\tilde{p}}^n = \frac{\partial \tilde{T}}{\partial q} - \frac{\partial U}{\partial q} - 2\lambda \begin{bmatrix} 0_3 \\ \eta \\ 0 \end{bmatrix} + \tilde{\tau}(t) \tag{E.7}$$

where λ is an (as yet undetermined) Lagrangian multiplier. The presence of a Lagrangian multiplier represents virtual forces necessary to satisfy the constraint (E.5).

Before we continue, we need to find $\partial\tilde{T}/\partial q$ and $\partial U/\partial q$.

Appendix E.2.1. The partial derivative $\partial\tilde{T}/\partial q$

From (B.13) we have that $\tilde{T}$ is not a function of x^n, but it is a function of η and q_t. Therefore,

$$\frac{\partial \tilde{T}}{\partial x^n} = 0. \tag{E.8}$$

Since M_t is not a function of η, we can use (Holden, 2011, Lemma A.3) to find that

$$\frac{\partial \tilde{T}}{\partial \eta} = \frac{1}{2}\frac{\partial(\dot{q}^\top P^\top M_t P\dot{q})}{\partial \eta} = \frac{1}{2}\frac{\partial(f^\top(q,\dot{q})M_t f(q,\dot{q}))}{\partial \eta} = \frac{\partial f}{\partial \eta}M_t f(q,\dot{q}) = \frac{\partial f}{\partial \eta}M_t P\dot{q}$$

[10] See Goldstein et al. (2002); Lanczos (1970) for more details as to why this is necessary.

with

$$f(q,\dot{q}) \triangleq P\dot{q} = \begin{bmatrix} R^\top \dot{x}^n \\ 2G\dot{\eta} \\ q_t \end{bmatrix} = \begin{bmatrix} R^\top \dot{x}^n \\ -2\dot{G}\eta \\ q_t \end{bmatrix} \in \mathbb{R}^{7\times 1}.$$

since $G\dot{\eta} = -\dot{G}\eta$ (by Property 5). The partial derivative $\partial f/\partial \eta$ is then given by

$$\begin{aligned} \frac{\partial f}{\partial \eta} &= \begin{bmatrix} \frac{\partial (R^\top \dot{x}^n)}{\partial \eta} & -2\frac{\partial (\dot{G}\eta)}{\partial \eta} & \frac{\partial q_t}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) & -2\frac{\partial \eta}{\partial \eta}(1 \otimes \dot{G}^\top) & 0_{4\times 1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) & -2\dot{G}^\top & 0_{4\times 1} \end{bmatrix} \end{aligned}$$

where (Holden, 2011, Lemma A.2) has been used and $\otimes$ is the Kronecker product.

Furthermore, $R^\top = \mathbb{I}_3 - 2\eta_r S(\eta_i) + 2S^2(\eta_i)$, giving

$$\frac{\partial R^\top}{\partial \eta} = -2\frac{\partial}{\partial \eta}(\eta_r S(\eta_i)) + 2\frac{\partial}{\partial \eta} S^2(\eta_i)$$

with (by (Holden, 2011, Lemma A.2))

$$\begin{aligned} \frac{\partial}{\partial \eta}(\eta_r S(\eta_i)) &= \frac{\partial \eta_r \mathbb{I}_3}{\partial \eta}(S(\eta_i)\otimes \mathbb{I}_3) + \frac{\partial S(\eta_i)}{\partial \eta}(\mathbb{I}_3 \otimes n_r \mathbb{I}_3) = \begin{bmatrix} (\operatorname{vec} S(\eta_i))^\top \\ 0_{3\times 9} \end{bmatrix} + \eta_r \frac{\partial S(\eta_i)}{\partial \eta} \\ \frac{\partial}{\partial \eta} S^2(\eta_i) &= \frac{\partial S(\eta_i)}{\partial \eta}(S(\eta_i)\otimes \mathbb{I}_3) - \frac{\partial S(\eta_i)}{\partial \eta}(\mathbb{I}_3 \otimes S(\eta_i)) \end{aligned}$$

where

$$\frac{\partial S(\eta_i)}{\partial \eta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives

$$\begin{aligned} \frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) &= -2\begin{bmatrix} (\operatorname{vec} S(\eta_i))^\top \\ 0_{3\times 9} \end{bmatrix}(\dot{x}^n \otimes \mathbb{I}_3) - 2\frac{\partial S(\eta_i)}{\partial \eta}[\eta_r \mathbb{I}_9 - S(\eta_i)\otimes \mathbb{I}_3 + \mathbb{I}_3 \otimes S(\eta_i)](\dot{x}^n \otimes \mathbb{I}_3) \\ &= -2\begin{bmatrix} [S(\eta_i)\dot{x}^n]^\top \\ \eta_r S(\dot{x}^n) - S(S(\eta_i)\dot{x}^n) + S(\dot{x}^n)S(\eta_i) \end{bmatrix} \\ &= -2\begin{bmatrix} \eta_i^\top S(\dot{x}^n) \\ \eta_r S(\dot{x}^n) - S(\eta_i)S(\dot{x}^n) + 2S(\dot{x}^n)S(\eta_i) \end{bmatrix}. \end{aligned} \tag{E.9}$$

Thus,

$$\frac{\partial \tilde{T}}{\partial \eta} = \begin{bmatrix} \frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) & -2\dot{G}^\top & 0_{4\times 1} \end{bmatrix} M_t P\dot{q}. \tag{E.10}$$

The product $P\dot{q}$ is not a function of q_t, but M_t is. We therefore find (by (Holden, 2011, Corollary A.4))

$$\begin{aligned} \frac{\partial \tilde{T}}{\partial q_t} &= \frac{1}{2}\dot{q}^\top P^\top \frac{\partial M_t}{\partial q_t} P\dot{q} = \frac{1}{2}\tilde{p}^\top \mathcal{P}^\top M_t^{-1}\mathcal{P}P^\top \frac{\partial M_t}{\partial q_t} P\mathcal{P}^\top M_t^{-1}\mathcal{P}\tilde{p} \\ &= \frac{1}{2}\tilde{p}^\top \mathcal{P}^\top M_t^{-1} \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & \frac{\partial m_{v,\dot{q}_t}}{\partial q_t} \\ \frac{\partial M_\nu}{\partial q_t} & \frac{\partial M_\omega}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{77}}{\partial q_t} \end{bmatrix} M_t^{-1}\mathcal{P}\tilde{p} \end{aligned}$$

since $\mathcal{P}P^\top = P\mathcal{P}^\top = \mathbb{I}_7$ (Properties 13 and 14).

From (B.6)–(B.12), we find that

$$\frac{\partial M_\nu}{\partial q_t} = \rho_t A(\varsigma_p)\frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t}S(r_t^b(\varsigma_p)) + \rho_t A(\varsigma_s)\frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t}S(r_t^b(-\varsigma_s)) = \rho_t A_0 S(r_t^b(\varsigma_p) - r_t^b(-\varsigma_s)) \tag{E.11}$$

$$\frac{\partial m_{v,\dot{q}_t}}{\partial q_t} = \rho_t A_0 \frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t}\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\varsigma_p) + \rho_t A_0 \frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t}\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(-\varsigma_s) = \rho_t A_0^2\left[\frac{1}{A(\varsigma_p)}\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\varsigma_p) - \frac{1}{A(\varsigma_s)}\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(-\varsigma_s)\right] \tag{E.12}$$

$$\begin{aligned}\frac{\partial M_\omega}{\partial q_t} &= -\rho_t \frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t}A(\varsigma_p)S^2(r_t^b(\varsigma_p)) - \rho_t\frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t}A(\varsigma_s)S^2(r_t^b(-\varsigma_s))\\ &= -\rho_t A_0\left[S^2(r_t^b(\varsigma_0)) - S^2(r_t^b(-\varsigma_s))\right]\end{aligned} \tag{E.13}$$

$$\begin{aligned}\frac{\partial m_{\omega,\dot{q}_t}}{\partial q_t} &= \rho_t A_0\frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t}S(r_t^b(\varsigma_p))\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\varsigma_p) + \rho_t A_0 \frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t}S(r_t^b(-\varsigma_s))\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(-\varsigma_s)\\ &= \rho_t A_0^2\left[\frac{1}{A(\varsigma_p)}S(r_t^b(\varsigma_p))\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(\varsigma_p) - \frac{1}{A(\varsigma_s)}S(r_t^b(-\varsigma_s))\frac{\mathrm{d}\bar{r}_t^b}{\mathrm{d}\sigma}(-\varsigma_s)\right]\end{aligned} \tag{E.14}$$

$$\frac{\partial m_{77}}{\partial q_t} = \rho_t\frac{A_0^2}{A(\varsigma_p)}\frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t} + \rho_t\frac{A_0^2}{A(\varsigma_s)}\frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t} = \rho_t A_0^3\left[\frac{1}{A^2(\varsigma_p)} - \frac{1}{A^2(\varsigma_s)}\right] \tag{E.15}$$

by differentiating under the integral sign. We note the partial derivatives of M_t are exclusively functions of q_t.

We can then write

$$\frac{\partial\tilde{T}}{\partial q_t} = \frac{1}{2}\tilde{p}^\top\mathcal{P}^\top M_t^{-1}\frac{\partial M_t}{\partial q_t}M_t^{-1}\mathcal{P}\tilde{p}. \tag{E.16}$$

We see that we can write $\partial\tilde{T}/\partial q$ as

$$\frac{\partial\tilde{T}}{\partial q} = \begin{bmatrix} 0_{3\times 7} \\ \begin{bmatrix} \frac{\partial R^\top}{\partial\eta}(\dot{x}^n\otimes\mathbb{I}_3) & -2\dot{G}^\top & 0_{4\times 1}\end{bmatrix} \\ \frac{1}{2}\dot{q}^\top P^\top\frac{\partial M_t}{\partial q_t}\end{bmatrix} M_t P\dot{q}. \tag{E.17}$$

Appendix E.2.2. The partial derivative $\partial U/\partial q$

The partial derivative of U with respect to q can be found by noting that it is not a function of x^n, but it is a function of η and q_t. We find

$$\frac{\partial U}{\partial x^n} = 0_{3\times 1} \tag{E.18}$$

$$\frac{\partial U}{\partial\eta} = -g\frac{\partial(e_z^\top R)}{\partial\eta}\left[mr_g^b + \rho_t\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right] \tag{E.19}$$

$$\frac{\partial U}{\partial q_t} = -g\rho_t e_z^\top R\left[r_t^b(\varsigma_p)A(\varsigma_p)\frac{\mathrm{d}\varsigma_p}{\mathrm{d}q_t} + r_t^b(-\varsigma_s)A(\varsigma_s)\frac{\mathrm{d}\varsigma_s}{\mathrm{d}q_t}\right] = -g\rho_t A_0 e_z^\top R\left[r_t^b(\varsigma_p) - r_t^b(-\varsigma_s)\right] \tag{E.20}$$

using differentiation under the integral.

From (8), we have

$$e_z^\top R = \begin{bmatrix} 2(\eta_{i,1}\eta_{i,3} - \eta_{i,2}\eta_r) & 2(\eta_{i,2}\eta_{i,3} + \eta_{i,1}\eta_r) & 1 - 2(\eta_{i,1}^2 + \eta_{i,2}^2)\end{bmatrix},$$

such that

$$\frac{\partial e_z^\top R}{\partial\eta} = 2\begin{bmatrix} -\eta_{i,2} & \eta_{i,1} & 0 \\ \eta_{i,3} & \eta_r & -2\eta_{i,1} \\ -\eta_r & \eta_{i,3} & -2\eta_{i,2} \\ \eta_{i,1} & \eta_{i,2} & 0\end{bmatrix} \in\mathbb{R}^{4\times 3}. \tag{E.21}$$

Appendix E.3. Using generalized momentum in $\mathbb{R}^7$

We define the body-fixed momentum vector p as

$$p \triangleq M_t \nu \in \mathbb{R}^7 \tag{E.22}$$

and note that $\tilde{p} = P^\top M_t P \dot{q} = P^\top M_t \nu = P^\top p$ which implies $p = \mathcal{P}\tilde{p}$ since $\mathcal{P}P^\top = \mathbb{I}_7$ (Property 13). Also worth noting is that

$$p = M_t P \dot{q} \tag{E.23}$$

$$\dot{q} = \mathcal{P}^\top M_t^{-1} p \tag{E.24}$$

since $P\mathcal{P}^\top = \mathbb{I}_7$ (Property 14).

Therefore,

$$\begin{aligned}
\dot{p} = \dot{\mathcal{P}}\tilde{p} + \mathcal{P}\dot{p}^n = \dot{\mathcal{P}}P^\top p + \mathcal{P}\dot{p}^n &= \begin{bmatrix} -S(\omega^b)R^\top & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & \frac{1}{2}\dot{G} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times4} & 0 \end{bmatrix} \begin{bmatrix} R & 0_{3\times3} & 0_{3\times1} \\ 0_{4\times3} & 2G^\top & 0_{4\times1} \\ 0_{1\times3} & 0_{1\times3} & 1 \end{bmatrix} p + \mathcal{P}\dot{p}^n \\
&= \begin{bmatrix} -S(\omega^b) & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & -\frac{1}{2}S(\omega^b) & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0 \end{bmatrix} p + \mathcal{P}\dot{p}^n
\end{aligned} \tag{E.25}$$

since $R^\top = R^{-1}$, $\dot{R} = RS(\omega^b)$ (Property 1) and $2\dot{G}G^\top = -S(\omega^b)$ (Property 7).

Inserting (E.7) into (E.25) gives

$$\begin{aligned}
\dot{p} &= \begin{bmatrix} -S(\omega^b) & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & -\frac{1}{2}S(\omega^b) & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0 \end{bmatrix} p + \mathcal{P}\left(\frac{\partial \tilde{T}}{\partial q} - \frac{\partial U}{\partial q} - 2\lambda \begin{bmatrix} 0_{3\times1} \\ \eta \\ 0 \end{bmatrix} + \tilde{\tau}(t) \right) \\
&= \begin{bmatrix} -S(\omega^b) & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & -\frac{1}{2}S(\omega^b) & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0 \end{bmatrix} p + \mathcal{P}\frac{\partial \tilde{T}}{\partial q} - \mathcal{P}\frac{\partial U}{\partial q} - 2\lambda\mathcal{P} \begin{bmatrix} 0_{3\times1} \\ \eta \\ 0 \end{bmatrix} + \mathcal{P}\tilde{\tau}(t).
\end{aligned} \tag{E.26}$$

We se that

$$\mathcal{P}\begin{bmatrix} 0_{3\times 1} \\ \eta \\ 0 \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} \\ \frac{1}{2}G\eta \\ 0 \end{bmatrix} = 0 \tag{E.27}$$

$$\mathcal{P}\tilde{\tau}(t) = \begin{bmatrix} R^\top \tau_f^n(t) \\ GG^\top \tau_m^b(t) \\ \tau_t(t) \end{bmatrix} = \begin{bmatrix} \tau_f^b(t) \\ \tau_m^b(t) \\ \tau_t(t) \end{bmatrix} \triangleq \tau(t) \tag{E.28}$$

$$\mathcal{P}\frac{\partial U}{\partial q} = \begin{bmatrix} R^\top \frac{\partial U}{\partial x^n} \\ \frac{1}{2}G\frac{\partial U}{\partial \eta} \\ \frac{\partial U}{\partial q_t} \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} \\ \frac{1}{2}G\frac{\partial U}{\partial \eta} \\ \frac{\partial U}{\partial q_t} \end{bmatrix} \tag{E.29}$$

$$\begin{aligned}
\mathcal{P}\frac{\partial \tilde{T}}{\partial q} &= \begin{bmatrix} 0_{3\times 7} \\ \frac{1}{2}G\begin{bmatrix} \frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) & -2\dot{G}^\top & 0_{4\times 1} \end{bmatrix} \\ \frac{1}{2}\dot{q}^\top P^\top \frac{\partial M_t}{\partial q_t} M_t^{-1} \end{bmatrix} M_t P \dot{q} \\
&= \begin{bmatrix} 0_{3\times 7} \\ \begin{bmatrix} \frac{1}{2}G\frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) & -G\dot{G}^\top & 0_{4\times 1} \end{bmatrix} \\ \frac{1}{2}p^\top M_t^{-1} \frac{\partial M_t}{\partial q_t} M_t^{-1} \end{bmatrix} p.
\end{aligned} \tag{E.30}$$

By using (7) and (E.9),

$$\begin{aligned}
G\frac{\partial R^\top}{\partial \eta}(\dot{x}^n \otimes \mathbb{I}_3) &= -2\begin{bmatrix} -\eta_i & \eta_r \mathbb{I}_3 - S(\eta_i) \end{bmatrix} \begin{bmatrix} \eta_i^\top S(\dot{x}^n) \\ \eta_r S(\dot{x}^n) - S(\eta_i)S(\dot{x}^n) + 2S(\dot{x}^n)S(\eta_i) \end{bmatrix} \\
&= 2\left[\eta_i\eta_i^\top - S^2(\eta_i) - \eta_r^2\mathbb{I}_3 + 2\eta_r S(\eta_i)\right] S(\dot{x}^n) + 4\left[S(\eta_i) - \eta_r\mathbb{I}_3\right] S(\dot{x}^n)S(\eta_i) \\
&= 2\left[\left(\eta_i^\top\eta_i - \eta_r^2\right)\mathbb{I}_3 + 2\eta_r S(\eta_i)\right] S(\dot{x}^n) + 4\left[S(\eta_i) - \eta_r\mathbb{I}_3\right] S(\dot{x}^n)S(\eta_i) \\
&= -2\big[S((\eta_r^2 - \eta_i^\top\eta_i)\dot{x}^n) + 2S(S^2(\eta_i)\dot{x}^n) - S(2\eta_r S(\eta_i)\dot{x}^n) + S(2\eta_i^\top\eta_i\dot{x}^n)\big] \\
&= -2S((\mathbb{I}_3 - 2\eta_r S(\eta_i) + 2S^2(\eta_i))\dot{x}^n) = -2S(R^\top\dot{x}^n) = -2S(v^b).
\end{aligned}$$

Furthermore, by Property 7, $2G\dot{G}^\top = S(\omega^b)$. From (E.19) we have

$$G\frac{\partial U}{\partial \eta} = -gG\frac{\partial e_z^\top R}{\partial \eta}\left[mr_g^b + \rho_t \int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right]. \tag{E.31}$$

From (7) and (E.21) we get

$$\begin{aligned}
G\frac{\partial e_z^\top R}{\partial \eta} &= 2\begin{bmatrix} -\eta_{i,1} & \eta_r & \eta_{i,3} & -\eta_{i,2} \\ -\eta_{i,2} & -\eta_{i,3} & \eta_r & \eta_{i,1} \\ -\eta_{i,3} & \eta_{i,2} & -\eta_{i,1} & \eta_r \end{bmatrix} \begin{bmatrix} -\eta_{i,2} & \eta_{i,1} & 0 \\ \eta_{i,3} & \eta_r & -2\eta_{i,1} \\ -\eta_r & \eta_{i,3} & -2\eta_{i,2} \\ \eta_{i,1} & \eta_{i,2} & 0 \end{bmatrix} \\
&= 2\begin{bmatrix} 0 & 1 - 2\eta_{i,1}^2 - 2\eta_{i,2}^2 & -2\eta_r\eta_{i,1} - 2\eta_{i,2}\eta_{i,3} \\ -1 + 2\eta_{i,1}^2 + 2\eta_{i,2}^2 & 0 & -2\eta_r\eta_{i,2} + 2\eta_{i,1}\eta_{i,3} \\ 2\eta_r\eta_{i,1} + 2\eta_{i,2}\eta_{i,3} & 2\eta_r\eta_{i,2} - 2\eta_{1,2}\eta_{i,3} & 0 \end{bmatrix} \\
&= -2S(R^\top e_z)
\end{aligned} \tag{E.32}$$

where we have used that $\eta^\top\eta = 1$.

Inserting the above into (E.26), we get

$$\begin{aligned}
\dot{p} &= -\begin{bmatrix} \begin{bmatrix} S(\omega^b) & 0_{3\times3} & 0_{3\times1} \end{bmatrix} \\ \begin{bmatrix} S(v^b) & S(\omega^b) & 0_{3\times1} \end{bmatrix} \\ \frac{1}{2}p^\top M_t^{-1}\frac{\partial M_t}{\partial q_t}M_t^{-1} \end{bmatrix} p - \begin{bmatrix} 0_{3\times1} \\ gS(R^\top e_z)\left[mr_g^b + \rho_t\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right] \\ -g\rho_t A_0 e_z^\top R\left(r_t^b(\varsigma_p) - r_t^b(-\varsigma_s)\right) \end{bmatrix} + \tau(t) \\
&= \tau(t) - \begin{bmatrix} \begin{bmatrix} \mathcal{S}(\nu) & 0_{6\times1} \end{bmatrix} \\ -\frac{1}{2}p^\top M_t^{-1}\frac{\partial M_t}{\partial q_t}M_t^{-1} \end{bmatrix} p - k(q) \qquad \text{(E.33)}
\end{aligned}$$

where

$$\mathcal{S}(\nu) = \begin{bmatrix} S(\omega^b) & 0_{3\times3} \\ S(v^b) & S(\omega^b) \end{bmatrix}$$

$$k_t(q) \triangleq g\begin{bmatrix} 0_{3\times1} \\ S(R^\top e_z)\left[mr_g^b + \rho_t\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} r_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right] \\ -\rho_t A_0 e_z^\top R\left[r_t^b(\varsigma_p) - r_t^b(-\varsigma_s)\right] \end{bmatrix} = \mathcal{P}\frac{\partial U}{\partial q} \qquad \text{(E.34)}$$

We define

$$p_l \triangleq \begin{bmatrix} \mathbb{I}_3 & 0_{3\times3} & 0_{3\times1} \end{bmatrix} p \in \mathbb{R}^3 \qquad \text{(E.35)}$$

$$p_r \triangleq \begin{bmatrix} 0_{3\times3} & \mathbb{I}_3 & 0_{3\times1} \end{bmatrix} p \in \mathbb{R}^3 \qquad \text{(E.36)}$$

$$p_t \triangleq \begin{bmatrix} 0_{1\times3} & 0_{1\times3} & 1 \end{bmatrix} p \in \mathbb{R} \qquad \text{(E.37)}$$

This allows us to rewrite the product $\begin{bmatrix} \mathcal{S}(\nu) & 0_{6\times1} \end{bmatrix} p$ in a more useful form:

$$\begin{aligned}
\begin{bmatrix} \mathcal{S}(\nu) & 0_{6\times1} \end{bmatrix} p &= \begin{bmatrix} \begin{bmatrix} S(\omega^b) & 0_{3\times3} \\ S(v^b) & S(\omega^b) \end{bmatrix} \begin{bmatrix} p_l \\ p_r \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0_{3\times3} & S(p_l) \\ S(p_l) & S(p_r) \end{bmatrix} \begin{bmatrix} v^b \\ \omega^b \end{bmatrix} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} C_s(p) & 0_{6\times1} \end{bmatrix} \nu = \begin{bmatrix} C_s(p) & 0_{6\times1} \end{bmatrix} M_t^{-1} p \qquad \text{(E.38)}
\end{aligned}$$

where

$$C_s(p) \triangleq -\begin{bmatrix} 0_{3\times3} & S(p_l) \\ S(p_l) & S(p_r) \end{bmatrix} = -C_s^\top(p). \qquad \text{(E.39)}$$

Similarly, the product $p^\top M_t^{-1}\frac{\partial M_t}{\partial q_t}M_t^{-1}p$ can be rewritten in a more useful form:

$$\begin{aligned}
\begin{bmatrix} 0_{3\times 1} \\ 0_{3\times 1} \\ \frac{1}{2}p^\top M_t^{-1}\frac{\partial M_t}{\partial q_t}M_t^{-1}p \end{bmatrix} &= \dot{q}_t \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times 1} \\ 0_{3\times 3} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & 0_{3\times 1} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t} \end{bmatrix} \nu \\
&\quad + \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & \frac{\partial M_\nu}{\partial q_t}\omega^b \\ 0_{3\times 3} & 0_{3\times 3} & -\frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} & \frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0 \end{bmatrix} \nu \\
&= \dot{q}_t \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times 1} \\ 0_{3\times 3} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & 0_{3\times 1} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t} \end{bmatrix} M_t^{-1}p \\
&\quad + \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & \frac{\partial M_\nu}{\partial q_t}\omega^b \\ 0_{3\times 3} & 0_{3\times 3} & -\frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} & \frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0 \end{bmatrix} M_t^{-1}p.
\end{aligned} \tag{E.40}$$

We can rewrite the kinematics as

$$\dot{q} = \mathcal{W}\tilde{p} = \mathcal{P}^\top M_t^{-1}\mathcal{P}P^\top p = \mathcal{P}^\top M_t^{-1}p \tag{E.41}$$

since $\mathcal{P}P^\top = \mathbb{I}_7$. Note that $M_t^{-1}p = \nu$, that is, the vector of generalized velocities in the body frame.

Noting that $\dot{q}_t = [0_{1\times 6}, 1]M_t^{-1}p$ and $\omega^b = [0_{3\times 3}, \mathbb{I}_3, 0_{3\times 1}]\, M_t^{-1}p$, we define

$$\begin{aligned}
C_t(q,p) &\triangleq \begin{bmatrix} \begin{matrix} & C_s & \end{matrix} & \begin{matrix} -\frac{\partial M_\nu}{\partial q_t}\omega^b \\ \frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \end{matrix} \\ -\ \omega^{b\top}\frac{\partial M_\nu}{\partial q_t} \quad -\frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0 \end{bmatrix} - \dot{q}_t \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times 1} \\ 0_{3\times 3} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & 0_{3\times 1} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t} \end{bmatrix} \\
&= \bar{C}_w - \dot{q}_t\bar{C}_m
\end{aligned} \tag{E.42}$$

were the function arguments have been omitted and

$$\bar{C}_w \triangleq \begin{bmatrix} \begin{matrix} & C_s & \end{matrix} & \begin{matrix} -\frac{\partial M_\nu}{\partial q_t}\omega^b \\ \frac{1}{2}\frac{\partial M_\omega}{\partial q_t}\omega^b \end{matrix} \\ -\ \omega^{b\top}\frac{\partial M_\nu^\top}{\partial q_t} \quad -\frac{1}{2}\omega^{b\top}\frac{\partial M_\omega^\top}{\partial q_t} & 0 \end{bmatrix} = -C_w^\top \tag{E.43}$$

$$\bar{C}_m \triangleq \begin{bmatrix} 0_{3\times 3} & -\frac{\partial M_\nu}{\partial q_t} & 0_{3\times 1} \\ 0_{3\times 3} & \frac{1}{2}\frac{\partial M_\omega}{\partial q_t} & 0_{3\times 1} \\ \frac{\partial m_{v,\dot{q}_t}^\top}{\partial q_t} & \frac{\partial m_{\omega,\dot{q}_t}^\top}{\partial q_t} & \frac{1}{2}\frac{\partial m_{77}}{\partial q_t} \end{bmatrix} \tag{E.44}$$

We can then define the dynamics as

$$\dot{q} = \mathcal{P}^\top(q)M_t^{-1}(q)p \tag{E.45}$$

$$\dot{p} = \tau(t) - C_t(q,p)M_t^{-1}(q)p - k_t(q). \tag{E.46}$$

We recognize (E.45)–(E.46) as matching Proposition 3.

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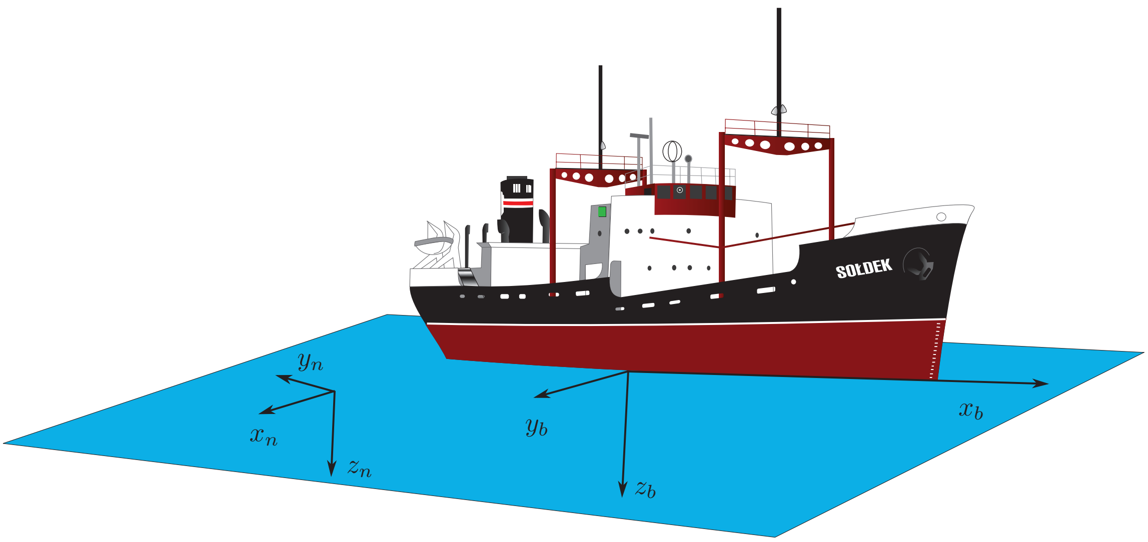

Figure 1: The body-fixed and inertial reference frames.

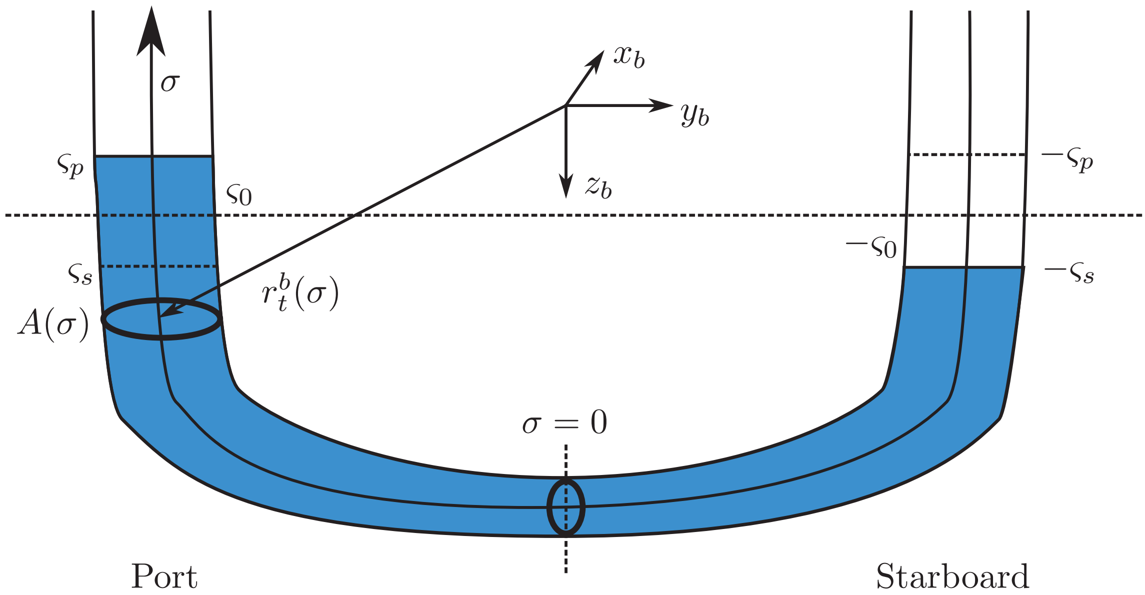

Figure 2: U-tank parameters.

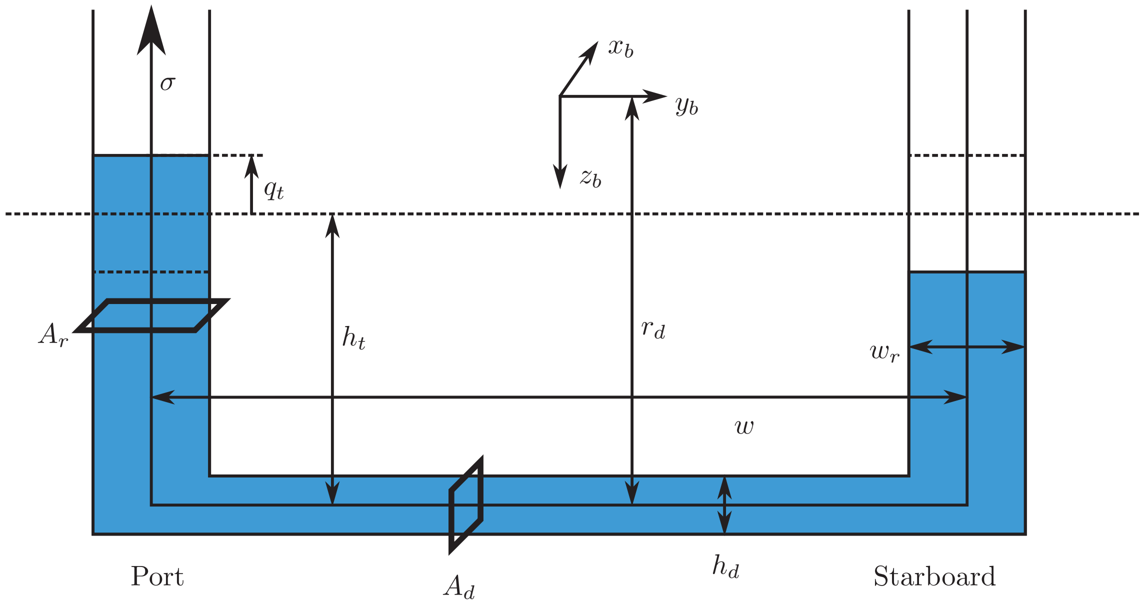

Figure 3: Measurements of the u-tank (63)–(66)