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$\mathcal{H}_\infty$ Almost Regulated Synchronization and $\mathcal{H}_\infty$ Almost Formation for Heterogeneous Networks under External Disturbances

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***Abstract*— We introduce the notions of "$\mathcal{H}_\infty$ almost regulated synchronization" and "$\mathcal{H}_\infty$ almost formation" for multi-agent systems subject to external disturbances and under directed interconnection structures. We assume that agents are linear, right-invertible and introspective with non-identical dynamics. The objective is to suppress the impact of disturbances on the synchronization error dynamics in terms of the $\mathcal{H}_\infty$ norm of the corresponding closed-loop transfer function. Inspired by the time-scale structure assignment techniques based on the singular perturbation theory, a family of observer-based protocols is introduced to achieve synchronization with any desired accuracy.**

I. INTRODUCTION

The problem of synchronization in networks of dynamic systems possesses diverse applications such as synchronization of coupled oscillators, distributed robotics, distributed sensor fusion in mobile sensor networks, quantum networks, and networked economics, to mention but a few.

A multi-agent system is referred to a group of dynamic systems (agents) that communicate with each other by message passing. The objective is state (output) synchronization; i.e. the states (outputs) of all agents have to agree on a common trajectory. Many works have studied networks of single-integrator and double-integrator systems and proposed static decentralized protocols. The common assumption among those works is that agents exchange relative state information; see e.g. [1], [2].

Synchronization in networks of identical linear agents of high orders is addressed in [3]–[6] where partial-state information is given to agents via the network; consequently, dynamic protocols are designed. A significant breakthrough in the design of dynamic protocols is presented in [4] where conventional observers are expanded to distributed observers assuming that agents are capable of exchanging information about the state of their own protocols over the network. The result is extended to LQR-based optimal design in [5]. Regulation of output synchronization is investigated in [6].

Synchronization in networks of non-identical agents has not been explored thoroughly yet. The common assumption is that agents are introspective; that is, agents possess some knowledge about their own states; see [7]–[10]. Output consensus for weakly minimum-phase systems of relative degree one is studied in [8] where local feedbacks are utilized to make the network partially identical. [9] proposes a controller for SISO minimum-phase systems by embedding models within each agents such that agent's output tracks the model's output. Using dynamic pre-compensators and local feedbacks, [10] puts forward a method to make a network of asymptotically identical agents. Relaxation of the self-knowledge (introspective) assumption, [11] introduces a distributed dynamic protocol for networks of linear, right-invertible agents. A thorough coverage of earlier work is found in [12] and [13] and references therein. A broad overview of recent progress in synchronization of multi-agent systems is provided in [14].

Contribution of The Paper [1]

The paper brings forth the notion of "$\mathcal{H}_\infty$ almost regulated synchronization[2]". We are interested in designing decentralized protocols for a network of introspective, right invertible agents with non-identical dynamics of any order subject to external disturbances and under directed interconnection topologies such that the regulation of the synchronization trajectory to a reference trajectory, which is generated by an exosystem, is achieved with *any* desired accuracy. In fact, we aim to construct a family of parameterized linear time-invariant protocols based on a distributed observer such that i) regulated output synchronization is accomplished in the absence of disturbances, and ii) the impact of disturbances on the regulation errors is attenuated to *any arbitrarily small* value in the sense of the $\mathcal{H}_\infty$ norm of the corresponding transfer function. The proposed method relies on time-scale structure assignment and is not iterative.

Moreover, we introduce the concept of "$\mathcal{H}_\infty$ almost formation" where agents are asked to maintain their relative outputs as desired while they are under the influence of disturbances. Moreover, we show that the results can be broadened to encompass another scenario called "$\mathcal{H}_\infty$ almost regulated formation" in which agents, under the influence of external disturbances, are to make a desired formation while they are asked to track a reference with a desired accuracy.

II. NOTATIONS AND PRELIMINARIES

Throughout the paper, matrix A is represented by $A = [a_{ij}]$ where the element (i,j) of A is shown by a_{ij}. The open

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[1] This paper is an extension to [15] where the notion of $\mathcal{H}_\infty$ *almost synchronization* is introduced for the first time.

[2] Since agents are non-identical, state synchronization is not meaningful; hence, the type of synchronization in this work is output synchronization.

left- (right-) half complex plane is represented by $\mathbb{C}^-(\mathbb{C}^+)$. A block diagonal matrix constructed by A_i's is represented by $\operatorname{diag}\{A_i\}$ for $i = 1, \cdots, n$. Also, $x = \operatorname{col}\{x_i\}$ for $i = 1, \cdots, n$ is adopted to denote $x = [x_1^{\mathrm{T}}, \cdots, x_n^{\mathrm{T}}]^{\mathrm{T}}$. The real part of a complex number λ, is represented by $\operatorname{Re}\{\lambda\}$. For a transfer function $T(s)$, the $\mathcal{H}_\infty$ norm is denoted $\|T(s)\|_\infty$.

Let $\mathcal{L}$ be a weighted directed graph with n nodes. If there is an edge from node j to node i, $a_{ij} > 0, a_{ij} \in \mathbb{R}$ is assigned to the edge; otherwise, $a_{ij} = 0$. If self-loops are not allowed, $a_{ii} = 0$. A directed graph has a directed spanning tree if there exists a node which has directed paths to every other nodes.

The Laplacian of $\mathcal{L}$ is denoted by $L = [l_{ij}]$ where $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. Thus, $\mathbf{1}_n$ is a right eigenvector of L associated with the eigenvalue at zero. If $\mathcal{L}$ contains a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues are in $\mathbb{C}^+$ [2].

III. Multi-Agent Systems

A multi-agent system is referred to a network of MIMO agents described by linear time-invariant models as

$$\text{Agent } i : \begin{cases} \dot{\bar{x}}_i = A_i \bar{x}_i + B_i \bar{u}_i + G_i \bar{w}_i & \text{(1a)} \\ y_i = C_i \bar{x}_i & \text{(1b)} \end{cases}$$

in which $i \in \mathbb{S} \triangleq \{1, 2, \cdots, N\}$, $\bar{x}_i \in \mathbb{R}^{n_i}$ is state, $\bar{u}_i \in \mathbb{R}^{m_i}$ is the input, $y_i \in \mathbb{R}^p$ is the output, $\bar{w}_i \in \mathbb{R}^{\omega_i}$: $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{w}_i^{\mathrm{T}} \bar{w}_i \,\mathrm{d}t < \infty$ is the disturbance. Agents are introspective; thus, each agent possesses partial knowledge of its own states as the local measurement[3]:

$$y_{m,i} = C_{m,i} \bar{x}_i \tag{1c}$$

Furthermore, agents are allowed to exchange information according to the network's communication topology which is described by a directed graph $\mathcal{L}$, with no self loops, associated with the adjacency matrix $A_{\mathcal{L}} = [a_{ij}]$ and the Laplacian matrix $L = [l_{ij}]$. In particular, the network measurement given to agent $i \in \mathbb{S}$ is:

$$\zeta_i = \sum\nolimits_{j=1}^{N} a_{ij}(y_i - y_j) \Rightarrow \zeta_i = \sum\nolimits_{j=1}^{N} l_{ij} y_j \tag{1d}$$

In addition, it is assumed that agents can exchange additional information over the network, which facilitates the design of distributed observers. Thus, agent i has access to:

$$\hat{\zeta}_i = \sum\nolimits_{j=1}^{N} l_{ij} \eta_j \tag{1e}$$

where $\eta_j \in \mathbb{R}^p$, depending on the state of protocol j, will be specified later.

Assumption 1: The following assumptions are made for each agent $i \in \mathbb{S}$.

(a) (A_i, B_i, C_i) is right-invertible; $(A_i, C_{m,i})$ is detectable;
(b) (A_i, B_i) is stabilizable and (A_i, C_i) is detectable;

A linear system is said to be right-invertible if there exists a rational matrix function of s, say $R(s)$, such that $H(s)R(s) = \mathrm{I}_p$ where $H(s)$ is the transfer matrix function of the system and p is the output dimension.

[3]This local measurement requires global sensing.

IV. $\mathcal{H}_\infty$ Almost Regulated Output Synchronization

In many applications, it is desirable to regulate the output of each agent, y_i, to a reference trajectory while synchronizing them. The section tackles this problem.

A. Problem Formulation

Consider a reference trajectory $y_0 \in \mathbb{R}^p$ which is defined as the output of an exosystem in the form

$$\bar{\Sigma}_0 : \begin{cases} \dot{\bar{x}}_0 = \bar{A}_0 \bar{x}_0 & \text{(2a)} \\ y_0 = \bar{C}_0 \bar{x}_0 & \text{(2b)} \end{cases}$$

where $\bar{x}_0 \in \mathbb{R}^{n_0}$, $(\bar{A}_0, \bar{C}_0)$ is observable, and $\bar{C}_0$ is full rank. For every agent, the regulation error is $\mathfrak{e}_{i,0} = y_i - y_0$. Define

$$\boldsymbol{w} \triangleq \operatorname{col}\{\bar{w}_i\}, \qquad \mathfrak{e}_{\mathbf{0}} = \operatorname{col}\{\mathfrak{e}_{i,0}\}, \qquad \forall i \in \mathbb{S}$$

Let $\mathfrak{e}_{\mathbf{0}} = T_{w\mathfrak{e}_0}(s)\boldsymbol{w}$. The problem of "$\mathcal{H}_\infty$ almost regulated (output) synchronization" is stated in Definition 1.

Definition 1: Consider the multi-agent system (1) with a communication topology $\mathcal{L}$. Given a set of network graphs $\mathcal{G}$ and any $\gamma > 0$, the problem of $\mathcal{H}_\infty$ *almost regulated synchronization* with respect to a reference y_0 evolved from (2) is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathcal{L} \in \mathcal{G}$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}_{\mathbf{0}}$ satisfies $\|T_{w\mathfrak{e}_0}(s)\|_\infty < \gamma$. ◀

Assumption 2: Every node of the network graph $\mathcal{L}$ is a member of a directed tree with the root contained in the "root set" $\pi \subset \mathbb{S}$.

Note that if the network graph is one connected component containing a directed spanning tree, the set π may only own one node which is the root of a directed spanning tree.

A certain subset of agents must know how far their outputs are from the reference y_0; otherwise, regulation is not possible. The set π contains those agents which receive $\mathfrak{e}_{i,0}$ via the network. It implies that the network measurement (1d) should be altered to

$$\widetilde{\zeta}_i = \sum\nolimits_{j=1}^{N} a_{ij}(y_i - y_j) + \psi_i(y_i - y_0) \tag{3}$$

where $\psi_i > 0$ if $i \in \pi$; otherwise, $\psi_i = 0$. Now, the exosystem is regarded as a new node, labeled with '0', and added to the network graph. The resulting graph is called the *augmented* network graph, denoted $\widetilde{\mathcal{L}}$, and is associated with the Laplacian matrix $\widetilde{L} = [\widetilde{l}_{ij}]$. Let $\psi = \operatorname{col}\{\psi_i\}$ and $\Psi = \operatorname{diag}\{\psi_i\}$ for $i \in \mathbb{S}$. Then, (3) is recast as

$$\widetilde{\zeta}_i = \sum_{j=0}^{N} \widetilde{l}_{ij} y_j, \qquad \widetilde{L} = \begin{bmatrix} 0 & 0 \\ -\psi & L + \Psi \end{bmatrix} \tag{4}$$

Assumption 2 ensures that the augmented graph $\widetilde{\mathcal{L}}$ has a directed spanning tree; see e.g. [11].

Definition 2: For given $\beta > 0$, an integer $N_0 \geq 1$, and an index set $\pi \subseteq \mathbb{S}$, $\mathcal{G}_{\beta,\pi}^{*}$ is the set of directed graphs composed of N nodes where $N \leq N_0$ such that every $\mathcal{L} \in \mathcal{G}_{\beta,\pi}^{*}$ satisfies Assumption 2 with the root set π, and the eigenvalues of the Laplacian of the augmented graph $\widetilde{\mathcal{L}}$, denoted $\widetilde{\lambda}_0, \widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_N$, satisfy $\operatorname{Re}\{\widetilde{\lambda}_i\} > \beta$ for $\widetilde{\lambda}_i \neq 0$. ◀

The additional information (1e) is adapted according to the augmented network topology and is given by

$$\widehat{\zeta}_i = \sum_{j=0}^{N} \widetilde{l}_{ij}\eta_j = \hat{\zeta}_i + \psi_i(\eta_i - \eta_0), \text{ for } i \in \mathbb{S} \quad (5)$$

where $\eta_j \in \mathbb{R}^p$ will be specified later. Agent '0', which is the exosystem, receives no information from the network; thus, $\widetilde{\zeta}_0 = \widehat{\zeta}_0 = 0$. Therefore, $\widehat{\zeta}_i = \sum_{j=0}^{N} \widetilde{l}_{ij}\eta_j$ for $i \in \{0\} \cup \mathbb{S}$. Note $\widetilde{\zeta}_i = 0, \forall i \in \mathbb{S}$, means simultaneous achievement of synchronization and output regulation: $y_1 = \cdots = y_N = y_0$.

B. Result 1: $\mathcal{H}_\infty$ Almost Regulated Output Synchronization

Theorem 1: Under Assumption 1 and for the set $\mathcal{G}^*_{\beta,\pi}$, the problem of $\mathcal{H}_\infty$ almost regulated output synchronization is solvable; specifically, there exists a family of linear time-invariant protocols, parameterized in terms of a tuning parameter $\epsilon \in (0,1]$, of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\epsilon)\chi_i + \mathcal{B}_i(\epsilon)\operatorname{col}\{\widetilde{\zeta}_i, \widehat{\zeta}_i, y_{m,i}\} & (6a)\\ \bar{u}_i = \mathcal{C}_i(\epsilon)\chi_i + \mathcal{D}_i(\epsilon)\operatorname{col}\{\widetilde{\zeta}_i, \widehat{\zeta}_i, y_{m,i}\} & (6b)\end{cases}$$

where $\chi_i \in \mathbb{R}^{q_i}$ and $i \in \mathbb{S}$ such that

(i) for any $\beta > 0$, there exists an $\epsilon_1^* \in (0,1]$ such that, for every $\epsilon \in (0, \epsilon_1^*]$, output regulation to the reference y_0 is accomplished in the absence of disturbance; i.e. $\forall\epsilon \in (0, \epsilon_1^*]$ when $\boldsymbol{w} = 0$

$$\mathfrak{e}_{i,0} = y_i - y_0 \to 0 \quad \text{as} \quad t \to \infty, \quad \forall i \in \mathbb{S}$$

(ii) for any given $\gamma > 0$, there exists an $\epsilon_2^* \in (0, \epsilon_1^*]$ such that, for every $\epsilon \in (0, \epsilon_2^*]$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}_0$ satisfies: $\|T_{w\mathfrak{e}_0}(s)\|_\infty < \gamma$. □

The proof is a direct consequence of Lemmas 1 and 2 and is given in a constructive way in the following subsections.

First, we make every agent mimic a desired (almost) identical dynamics; then, the protocol is designed for the resulting system.

C. Almost Identical Representation for Augmented Network

In this section, we show how the augmented network can be converted into a network of almost identical agents.

Definition 3: Let $n_{q0} \geq 1$ be the maximum order of the infinite zeros of all triples (A_i, B_i, C_i), $i \in \mathbb{S}$. Let n^*_{q0} be the largest observability index of the pair $(\bar{A}_0, \bar{C}_0)$. ◀

There exist a series of state manipulations for the exosystem (2) and a matrix B_0 such that the resulting system characterized by the triple (A_0, B_0, C_0) is invertible, of the uniform rank $n_q \geq n^*_{q0}$, with no invariant zeros (see Appendix I). Then, according to [16], there exists a nonsingular state transformation such that the system described by the triple (A_0, B_0, C_0) is represented in the following form:

$$\Sigma_0 : \begin{cases} \dot{x}_0 = Ax_0 + B(M_0u_0 + R_0x_0) & (7a)\\ y_0 = Cx_0 & (7b)\end{cases}$$

where $x_0 \in \mathbb{R}^{pn_q}$, $u_0 \in \mathbb{R}^p$, $R_0 \in \mathbb{R}^{p\times pn_q}$ while $M_0 \in \mathbb{R}^{p\times p}$ is nonsingular. The matrices A, B and C are given by

$$A = \begin{bmatrix} 0 & \mathrm{I}_{p(n_q-1)} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \mathrm{I}_p \end{bmatrix}, C = \begin{bmatrix} \mathrm{I}_p & 0 \end{bmatrix} \quad (8)$$

Since the exosystem is autonomous and we have no control over it, u_0 must be zero.

We adopt a four-step procedure as [17] in order to shape a multi-agent system of the form (1) into the desired form (12) where agents are partially identical. The method requires rank-equalizing and squaring-down pre-compensators as well as local dynamic feedbacks within each agent. The shaping is the key element for solvability of the problem. The requisite for shaping is the local measurement (1c).

Lemma 1: Consider the multi-agent system (1) satisfying Assumption 1-(a). Let $n_q \geq n_{q0}$. For each agent, there exists a local dynamic compensator

$$\begin{cases} \dot{\bar{\tilde{x}}}_i = \widetilde{A}_i\bar{\tilde{x}}_i + \widetilde{B}_{1i}u_i + \widetilde{B}_{2i}y_{m,i} & (9a)\\ \bar{u}_i = \widetilde{C}_i\bar{\tilde{x}}_i + \widetilde{D}_iu_i + \widetilde{D}_{2i}y_{m,i} & (9b)\end{cases}$$

such that the application of (9) to (1) can be written as

$$\begin{cases} \dot{x}_i = Ax_i + B(Mu_i + Rx_i) + E_{d,i}\bar{w}_i + \rho_i & (10a)\\ y_i = Cx_i & (10b)\end{cases}$$

where $R \in \mathbb{R}^{p\times pn_q}$ and nonsingular $M \in \mathbb{R}^{p\times p}$ are selected arbitrarily while A, B and C are given by (8). In (10), $x_i \in \mathbb{R}^{pn_q}$, u_i, $y_i \in \mathbb{R}^p$. Also, ρ_i is evolved from the system described by

$$\begin{cases} \dot{\tilde{x}}_i = H_i\tilde{x}_i + E_{o,i}\bar{w}_i & (11a)\\ \rho_i = W_i\tilde{x}_i & (11b)\end{cases}$$

where H_i is Hurwitz stable.

Proof: See [17]. ■

Lemma 1 states that agents can be shaped into the dynamics of any invertible system with no invariant zeros and of uniform rank n_q. Therefore, R and M can be chosen arbitrarily. As H_i is Hurwitz stable, $\tilde{x}_i$ and ρ_i have the same nature as $\bar{w}_i$, and one can redefine external disturbances as $w_i \triangleq \operatorname{col}\{\bar{w}_i, \tilde{x}_i\}$. Hence, the system (10) is recast as

$$\begin{cases} \dot{x}_i = Ax_i + B(Mu_i + Rx_i) + E_iw_i & (12a)\\ y_i = Cx_i & (12b)\end{cases}$$

where $E_i = [E_{d,i},\ W_i]$. Redefining disturbance changes $\|T_{w\mathfrak{e}_0}(s)\|_\infty$; however, because the $\mathcal{H}_\infty$-norm of (11) is constant with respect to ϵ, it does not affect the solvability of the problem, and the $\mathcal{H}_\infty$ almost regulated synchronization problem can be achieved for any given $\gamma > 0$ by an appropriate choice of ϵ.

Therefore, for $n_q \geq \max\{n_{q0}, n^*_{q0}\}$, in accordance with Lemma 1, there exist local dynamic compensators that shape agents into (12) for particular R_0 and M_0 obtained from (7). Consequently, the augmented network can be represented as a network of almost identical agents described by (12) for $i \in \{0\} \cup \mathbb{S}$ where $M = M_0$, $R = R_0$ and $E_0 = 0$.

D. Dynamic Protocol

Suppose that the agents are described by (12) with the particular choice of R_0 and M_0. For every $i \in \widetilde{\mathbb{S}} \triangleq \{0\} \cup \mathbb{S}$, the following dynamic protocol is proposed:

$$\dot{\hat{x}}_i = A\hat{x}_i + B(M_0u_i + R_0\hat{x}_i) - \epsilon^{-1}K(\widetilde{\zeta}_i - \widehat{\zeta}_i) \quad (13a)$$

$$u_i = \epsilon^{-n_q}M_0^{-1}FS\hat{x}_i \quad (13b)$$

where $\hat{x}_i \in \mathbb{R}^{pn_q}$, and $\epsilon \in (0,1]$ is the tuning parameter; F and K are the gains to be specified. The quantity $\widehat{\zeta}_i$ is given by (5) where $\eta_j = C\hat{x}_j$. The matrix S is defined as

$$S \triangleq \operatorname{diag}\{\mathrm{I}_p, \epsilon\, \mathrm{I}_p, \cdots, \epsilon^{n_q-1}\mathrm{I}_p\} \in \mathbb{R}^{pn_q \times pn_q} \quad (14)$$

Notice that setting $\hat{x}_0(0) = 0$ leads to $\hat{x}_0(t) = 0$, $\eta_i(t) = 0$, and $u_0(t) = 0\ \forall t \geq 0$. Note that the proposed protocol has a fixed order. To choose the protocol gains, we need to partition the system matrices (8) as

$$A = \begin{bmatrix} 0_{p\times p} & C_1 \\ 0_{p(n_q-1)\times p} & A_1 \end{bmatrix}, B = \begin{bmatrix} 0_{p\times p} \\ B_1 \end{bmatrix}, E_i = \begin{bmatrix} E_{1,i} \\ E_{2,i} \end{bmatrix}$$

where $C_1 = [\mathrm{I}_p, 0, \cdots, 0] \in \mathbb{R}^{p\times p(n_q-1)}$ and $E_{1,i} \in \mathbb{R}^{p\times \omega_i}$. Also partition $R_0 = [R_1,\ \bar{R}]$ where $R_1 \in \mathbb{R}^{p\times p}$. Consider the following structure for the observer gain:

$$K = \begin{bmatrix} K_1 \\ \bar{K}K_1 \end{bmatrix} \quad \text{where} \quad \begin{array}{l} K_1 \in \mathbb{R}^{p\times p} \\ \bar{K} \in \mathbb{R}^{p(n_q-1)\times p} \end{array} \quad (15)$$

Design Procedure The protocol gains are chosen as follows:

- considering the controllable pair (A, B), choose F such that $A + BF$ is Hurwitz stable;
- considering the observable pair $(A_1 + B_1\bar{R}, C_1)$, choose $\bar{K}$ such that $\widetilde{A}_z \triangleq A_1 + B_1\bar{R} - \bar{K}C_1$ is Hurwitz stable;
- choose $K_1 = K_1^{\mathrm{T}} < 0$.

Lemma 2: Consider that the exosystem is described by (7) where $u_0 = 0$. Consider a multi-agent system with a communication topology $\mathcal{L}$ and agents which are described by (12) where $R = R_0$ and $M = M_0$ as the exo-system. For any given $\gamma > 0$ and $\mathcal{G}^*_{\beta,\pi}$, the dynamic protocol (13) solves the problem of $\mathcal{H}_\infty$ almost regulated output synchronization for any $\mathcal{L} \in \mathcal{G}^*_{\beta,\pi}$.

Proof: See Appendix II. ■

E. Design Scheme

Given a multi-agent system of the form (1) with a communication topology $\mathcal{L} \in \mathcal{G}^*_{\beta,\pi}$, and an exosystem (2). The problem of $\mathcal{H}_\infty$ almost regulated synchronization is solved in the following fashion. i) Choose $n_q \geq \max\{n_{q0}, n^*_{q0}\}$. Represent the exosystem (2) in the form (7) to obtain R_0 and M_0. ii) According to Lemma 1, design local compensators within each agent so as to convert the multi-agent system to a network of almost identical agents (12) with $R = R_0$ and $M = M_0$. iii) Use the parameterized protocol (13). Find ϵ^*_1, and tune $\epsilon \in (0, \epsilon^*_1]$ to achieve a desired accuracy of output regulation.

It is evident that the structure of the protocols is independent of the parameter ϵ, and one can develop the protocol structure at one stage and tune the parameter $\epsilon \in (0, \epsilon^*_1]$ in order to reach the desired accuracy of output regulation. Since the structure is continuous in the parameter ϵ, tuning may be even carried out online. Notice, in terms of (1d) and (1e), eq. (13a) can be written as $\dot{\hat{x}}_i = A\hat{x}_i + B(M_0u_i + R_0\hat{x}_i) - \epsilon^{-1}K(\zeta_i - \hat{\zeta}_i) - \epsilon^{-1}K\psi_i(y_i - y_0 - C\hat{x}_i)$.

V. $\mathcal{H}_\infty$ Almost Formation

In formation control, the objective is to maintain the relative outputs among agents as desired. Let formation be defined in terms of a set of formation vectors $\mathbb{S}_f \triangleq \{f_1, \cdots, f_N\}$, $f_i \in \mathbb{R}^p$. Define $y_{f,i} = y_i - f_i$ for $i \in \mathbb{S}$. The mutual disagreement is denoted $\mathfrak{e}^*_{i,j} = y_{f,i} - y_{f,j}$ for $i, j \in \mathbb{S}, i > j$. The transfer function $T_{w\mathfrak{e}_f}(s)$ is defined as:

$$\mathfrak{e}_f \triangleq \operatorname{col}\{\mathfrak{e}^*_{i,j}\}, \qquad \mathfrak{e}_f = T_{w\mathfrak{e}_f}(s)\boldsymbol{w}$$

Definition 4: Consider the multi-agent system (1) with a communication topology $\mathcal{L}$. Given a set of network graphs $\mathcal{G}$ and any $\gamma > 0$, the problem of "$\mathcal{H}_\infty$ almost formation" with respect to a formation set $\mathbb{S}_f$ is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathcal{L} \in \mathcal{G}$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}_f$ satisfies $\|T_{w\mathfrak{e}_f}(s)\|_\infty < \gamma$. ◀

Formation is possible when agents exchange $y_{f,i}$'s. Therefore, the network information (1d) is to be modified to

$$\zeta_{f,i} = \zeta_i - \sum_{j=1}^{N} l_{ij}f_j = \sum_{j=1}^{N} l_{ij}y_{f,j} \quad (16)$$

The formation controller relies extensively on the fact that shaping is viable to any invertible system of uniform rank $n_q \geq n_{q0}$ which has no invariant zeros. Thus, Lemma 1 guarantees existence of local feedback laws to shape the system into the desired structure as (12) for arbitrary R and M ($\det(M) \neq 0$).

In view of (1) with agents described by (12), we propose the parameterized dynamic protocol for every agent $i \in \mathbb{S}$

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + B(Mu_i + R\hat{x}_i + R_1f_i) - \epsilon^{-1}K(\zeta_{f,i} - \hat{\zeta}_i) \\ u_i &= \epsilon^{-n_q}M^{-1}FS\hat{x}_i - M^{-1}R_1f_i \end{aligned} \quad (17)$$

where $\hat{x}_i \in \mathbb{R}^{pn_q}$ and $\epsilon \in (0,1]$ is the tuning parameter. The matrix S is as (14) and K is partitioned as (15). F, $\bar{K}$ and K_1 are chosen similar to the procedure presented in Subsection IV-D. The quantity $\zeta_{f,i}$ is found using (16). Considering $\eta_i = C\hat{x}_i$, $\hat{\zeta}_i$ is given by (1e).

Definition 5: For given $\beta > 0$ and integer $N_0 \geq 1$, $\mathcal{G}_\beta$ is the set of graphs composed of N nodes where $N \leq N_0$ such that every $\mathcal{L} \in \mathcal{G}_\beta$ has a directed spanning tree and the eigenvalues of its Laplacian, denoted $\lambda_1, \cdots, \lambda_N$, satisfy $\operatorname{Re}\{\lambda_i\} > \beta$ for every nonzero λ_i. ◀

The Laplacian matrix L associated with $\mathcal{L} \in \mathcal{G}_\beta$ has a simple eigenvalue at zero and the rest are located in $\mathbb{C}^+$; see [2]. Theorem 2 states the result formally.

Theorem 2: Under Assumption 1 and for the set $\mathcal{G}_\beta$, the problem of $\mathcal{H}_\infty$ almost formation is solvable; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter $\epsilon \in (0,1]$, of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\epsilon)\chi_i + \mathcal{B}_i(\epsilon)\ \operatorname{col}\{\zeta_{f,i}, \hat{\zeta}_i, y_{m,i}, f_i\} & (18a) \\ \bar{u}_i = \mathcal{C}_i(\epsilon)\chi_i + \mathcal{D}_i(\epsilon)\ \operatorname{col}\{\zeta_{f,i}, \hat{\zeta}_i, y_{m,i}, f_i\} & (18b) \end{cases}$$

where $\chi_i \in \mathbb{R}^{q_i}$ and $i \in \mathbb{S}$ such that

(i) for any $\beta > 0$, there exists an $\epsilon^*_1 \in (0,1]$ such that, for every $\epsilon \in (0, \epsilon^*_1]$, the desired formation is attained in the absence of disturbance; i.e. $\forall \epsilon \in (0, \epsilon_1^*]$, $\mathbf{e}_{i,j}^* \to 0$ as $t \to \infty$ when $\boldsymbol{w} = 0$.

(ii) for any given $\gamma > 0$, there exists an $\epsilon_2^* \in (0, \epsilon_1^*]$ such that, for every $\epsilon \in (0, \epsilon_2^*]$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathbf{e}_f$ satisfies $\|T_{w\mathbf{e}_f}(s)\|_\infty < \gamma$.

Proof: See Appendix III. ■

VI. Illustrative Example

The result is illustrated for a network consisting of $N = 4$ right-invertible agents with $p = 1$. The interconnection topology of the network is given by the digraph displayed in Fig. 1. The models of agents can be found in [17]. Disturbances are chosen $\bar{w}_1 = \sin(t)$, $\bar{w}_2 = 1$, $\bar{w}_4 = \sin(2t)$, and $\|\bar{w}_3\| \le 5$ which is a uniform random number. The order of the infinite zeros of agents are respectively 3, 2, 1, and 2. Thus, $n_{q0} = 3$. Network shaping is given in [17].

$\mathcal{H}_\infty$ Almost Regulated Output Synchronization: The consensus trajectory for the multi-agent system is desired to be $2\cos(\omega_0 t)$ generated by the exo-system (2) with

$$\bar{A}_0 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \quad \bar{C}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \bar{x}_0(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus, $n_q = n_{q0} = 3$. Let $\bar{B}_0 = [0,\ 1]^{\mathrm{T}}$. The triple $(\bar{A}_0, \bar{B}_0, \bar{C}_0)$ is invertible. A rank-equalizing pre-compensator is designed to change the relative degree to n_q. Transforming the system using the observability matrix renders the exosystem as (7) with $R_0 = \begin{bmatrix} 0, & -\omega_0^2, & 0 \end{bmatrix}$ and $M_0 = 1$ where $x_0 = [2,\ 0,\ -2\omega_0^2]^{\mathrm{T}}$. Agent 1 is the root of one spanning tree in the network graph, and we link agent 0 with agent 1 with a weight $\psi_1 = 1$. Let $\omega_0 = 0.5$. Fig. 2 displays the results, where the outputs of all agents are plotted, for $\epsilon = 0.01$ and $\epsilon = 0.05$. As expected, smaller ϵ leads to a more accurate output regulation.

$\mathcal{H}_\infty$ Almost Regulated Formation: This part builds up $\mathcal{H}_\infty$ almost regulated output synchronization into $\mathcal{H}_\infty$ almost formation. This can be portrayed as virtual reference formation control. Simulation is carried out for the same network as before. The formation set is selected as $\mathbb{S}_f = \{10, 5, -5, -10\}$. Thus, the objective is to have $y_1 - 10 = y_2 - 5 = y_3 + 5 = y_4 + 10 = y_0 \doteq 2\cos(0.5t)$ as $t \to \infty$. The result is given in Fig. 3.

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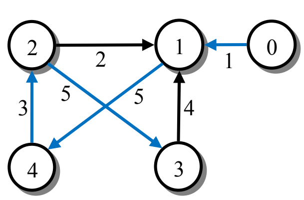

Fig. 1: Communication topology and the graph Laplacian. The original network consists of agents 1, 2, 3, and 4. Agent 0 is the exo-system.

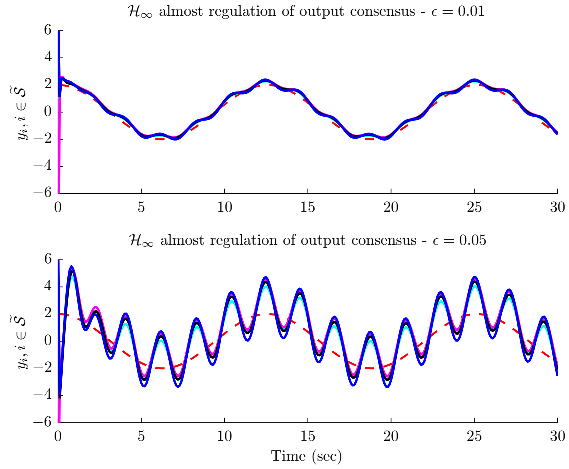

Fig. 2: The upper plot is for $\epsilon = 0.01$; the lower plot is for $\epsilon = 0.05$. The red dash lines show y_0 and the rest are y_i's.

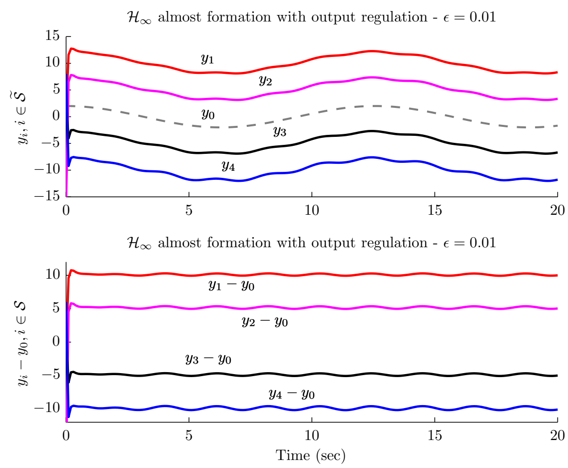

Fig. 3: The upper plot shows y_i, $i \in \widetilde{\mathbb{S}}$ and the lower plot depicts the regulation error $y_i - y_0$, $i \in \mathbb{S}$.

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APPENDIX I
MANIPULATION OF EXOSYSTEM

The system is of uniform rank n_q if all infinite zeros are of order n_q. Equivalently, if Markov parameters $M_{i+1} = CA^iB$ are zero for $i = 0, 1, \cdots, n_q - 2$ and $M_{n_q} = CA^{n_q-1}B$ is nonsingular. The method to convert an exosystem in the form of (2) into the form (7) is based on observability structural decomposition. Let $A_i \in \mathbb{R}^{k_i \times k_i}$, for $k_i \geq 1$, and $A_i^* \in \mathbb{R}^{n_q \times n_q}$ be as

$$A_i = \begin{bmatrix} 0 & \mathrm{I}_{k_i-1} \\ 0 & 0 \end{bmatrix}, \qquad A_i^* = \begin{bmatrix} 0 & \mathrm{I}_{n_q-1} \\ 0 & 0 \end{bmatrix} \tag{19}$$

For $k_i = 1$, $A_i = 0$. It follows from Theorem 4.3.1 in [18] that there exist nonsingular state and output transformations as $\bar{x}_0 = T_s \bar{z}_0$ and $y_0 = T_y \bar{y}_0$ such that the exosystem (2) can be described by p-interconnected systems in the form of:

$$\dot{\bar{z}}_i = A_i \bar{z}_i + L_i \bar{y}_0, \qquad \bar{z}_i \in \mathbb{R}^{k_i} \tag{20a}$$
$$\bar{y}_{0i} = C_i \bar{z}_i, \qquad \bar{y}_{0i} \in \mathbb{R}, \quad i = 1, \cdots, p \tag{20b}$$

in which $\bar{y}_0 = \mathrm{col}\{\bar{y}_{0i}\}$, $\bar{z}_0 = \mathrm{col}\{\bar{z}_i\}$, and k_i belongs to the set $\{k_1, \cdots, k_p\}$ which is the observability index of the pair $(\bar{A}_0, \bar{C}_0)$; $\sum_{j=1}^{p} k_j = n_0$. Also, $C_i = [1, 0, \cdots, 0] \in \mathbb{R}^{k_i}$ and $L_i \in \mathbb{R}^{k_i \times p}$, $i = 1, \cdots, p$, are some constant matrices.

We extend the dimension of each subsystem to n_q by adding an appropriate number of integrators to the bottom of $\bar{z}_i$. The initial values of the new states are set equal to zero. It does not affect $\bar{y}_{0i}$'s. The systems described by (20) are recast as

$$\dot{\tilde{z}}_i = A_i^* \tilde{z}_i + L_i^* \bar{y}_0, \quad \tilde{z}_i \in \mathbb{R}^{n_q}, \quad L_i^* = [L_i^{\mathrm{T}},\ 0]^{\mathrm{T}}$$
$$\bar{y}_{0i} = C_i^* \tilde{z}_i, \qquad C_i^* = [1, 0, \cdots, 0]$$

Define $L^* = \mathrm{stack}\{L_i^*\}$, $C^* = \mathrm{diag}\{C_i^*\}$, and $A_0 = \mathrm{diag}\{A_i^*\} + L^*C^*$. Let $B_i^* = [0, \cdots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^{n_q}$ and $B_0 = \mathrm{diag}\{B_i^*\}$. The triple (A_0, B_0, C^*) is invertible, of the uniform rank n_q with no invariant zeros. We restore the output to the original output by the output transformation T_y. Thus, let $C_0 = T_y C^*$. As invertibility is invariant under output transformation, the triple (A_0, B_0, C_0) is invertible, of the uniform rank n_q, with no invariant zeros.

APPENDIX II
PROOF OF LEMMA 2

Since the network graph $\mathcal{L}$ belongs to $\mathcal{G}^*_{\beta,\pi}$, the augmented graph $\widetilde{\mathcal{L}}$ has a directed spanning tree, which implies that it has one eigenvalue at zero with multiplicity one and $\mathrm{Re}\{\widetilde{\lambda}_i\} > \beta$ for $\widetilde{\lambda}_i \neq 0$. On the other hand, the agents are described by (12) with specific R_0 and M_0 which come from the modified exosystem described by (7); thus, all agents of the augmented network have almost identical dynamics. Therefore, the conditions of Lemma 1 in [17] are satisfied. From Lemma 1 in [17], it follows that synchronization is achieved in the absence of disturbance; i.e. $y_1 = \cdots = y_N = y_0$. It implies that regulation to the reference y_0 is accomplished.

In the presence of disturbances, it ensures that $\|T^{i,j}_{w\mathfrak{e}}\|_\infty < \epsilon\gamma_1'$ where $\mathfrak{e}_{i,j} = y_i - y_j$ for $i, j \in \widetilde{\mathbb{S}}$, $i > j$ and $\mathfrak{e}_{i,j} = T^{i,j}_{w\mathfrak{e}}(s)\boldsymbol{w}$. It is then viable to conclude that $\|T_{w\mathfrak{e}_0}\|_\infty < \epsilon\gamma_2'$ for some $\gamma_2' > 0$; thus, there exits an ϵ such that $\|T_{w\mathfrak{e}_0}\|_\infty < \gamma$.

It is worth noting the shaping provides an elegant way to solve the regulation problem, in which the regulator equation is trivially solved because the exosystem and the agents have the same (A, B, C).

APPENDIX III
PROOF OF THEOREM 2

It follows from Lemma 1 that there exists a dynamic compensator that makes agent $i \in \mathbb{S}$ have the dynamics of (12) for an arbitrary R and nonsingular M. Let $R \in \mathbb{R}^{p \times pn_q}$ be partitioned as $R = [R_1,\ \bar{R}]$ where $R_1 \in \mathbb{R}^{p \times p}$. The vector $\bar{f}_i \in \mathbb{R}^{pn_q}$ for $i \in \mathbb{S}$ is formed as $\bar{f}_i = \mathrm{col}\{f_i, 0\}$ such that $f_i = C\bar{f}_i$. Let $x_{f,i} = x_i - \bar{f}_i$. In view of the fact that $A\bar{f}_i = 0$, $R\bar{f}_i = R_1 f_i$ and $\dot{f}_i = 0$, (12) is recast as

$$\dot{x}_{f,i} = Ax_{f,i} + B(Mu_{f,i} + Rx_{f,i}) + E_i w_i \tag{22a}$$
$$y_{f,i} = Cx_{f,i} \tag{22b}$$

where $Mu_{f,i} = Mu_i + R_1 f_i$. In compliance with Section IV-D, the protocol for (22) will take the following form

$$\dot{\hat{x}}_i = A\hat{x}_i + B(Mu_{f,i} + R\hat{x}_i) - \epsilon^{-1}K\sum_{j=1}^{N} l_{ij}C\tilde{x}_j$$
$$u_{f,i} = \epsilon^{-n_q}M^{-1}FS\hat{x}_i \Rightarrow \quad u_i = \epsilon^{-n_q}FS\hat{x}_i - M^{-1}R_1 f_i$$

in which $\tilde{x}_j = x_{f,j} - \hat{x}_j$. Therefore, one may adopt Lemma 1 in [17] to attain the desired $\mathcal{H}_\infty$ gain since agents are in almost identical representation and $\mathcal{L} \in \mathcal{G}_\beta$. Lemma 1 in [17] states that $\lim_{t\to\infty}|y_{f,i} - y_{f,j}| = 0$ in the absence of disturbances. Hence, $\mathcal{H}_\infty$ almost formation is accomplished.