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$\mathscr{H}_\infty$ Almost Synchronization for Homogeneous Networks of Non-Introspective SISO Agents Under External Disturbances

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Abstract— **This paper addresses the problem of "$\mathscr{H}_\infty$ almost synchronization" for networks of identical, linear agents under directed communication topologies. Agents are presumed to be *non-introspective*; i.e. agents are not aware of their own state or output, and every agent is only provided with a linear combination of its own output relative to that of the neighbors. Providing the solvability conditions, a family of dynamic protocols is developed such that the impact of disturbances on the synchronization error dynamics, expressed in terms of the $\mathscr{H}_\infty$ norm of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value. Thus, synchronization with any desired accuracy can be achieved.**

I. Introduction

Consensus problem in networks of dynamic systems possesses diverse applications in various disciplines of science and engineering including synchronization of coupled oscillators, flocking, formation control, and distributed sensor fusion; see [1] and references therein. In consensus of networked complex systems, the objective is to find a distributed consensus algorithm to reach an agreement on a certain quantity of interest which depends on the states of agents.

The seminal works of [2] and [3] have substantially contributed in analysis and design of multi-agent systems by introducing the application of the graph theory and the Kronecker product. Moreover, the works of [4]–[7], and [8] have been instrumental in paving the way that consensus protocols have been developed. Also, [9] has extended conventional observers to distributed observers with the aid of allowing agents to exchange their protocol's states. A thorough coverage of earlier work, including static and dynamic protocols, the effect of communication delay, and dynamic interaction topologies may be found in [10] and [11] and references therein. A recent research on output synchronization of networks of heterogeneous agents is presented in [12].

A. The Topic of This Paper

In [13], we introduced the notion of $\mathscr{H}_\infty$ almost synchronization, in which the impact of external disturbances on disagreement dynamics is attenuated to 'any' arbitrarily small value in the sense of the $\mathscr{H}_\infty$ norm of the closed-loop transfer function; thus, synchronization can be achieved with any arbitrary accuracy. The work of Peymani et al. [13] is concerned with *heterogeneous* networks of "*introspective*" agents; i.e. the agents are non-identical, and have partial knowledge about their own state in addition to network measurements. In this article, the problem of $\mathscr{H}_\infty$ almost synchronization is solved for *homogeneous* networks of "*non-introspective*" agents; i.e. the network consists of identical agents which are *not* allowed to have access to their own states or outputs, and the only available measurement for each agent is a linear combination of its output relative to that of their neighbors. An example may be a fleet of underwater vehicles, the absolute position of any of which is not measured while each of them is aware of its relative position with respect to its neighbors.

We stress the fact that the lack of self-measurements does not allow us to shape the agents into the desired dynamics as we proposed in [13]. Hence, we are confronted with general linear systems, where the finite and infinite zero structures as well as invertibility properties of agents (in case agents are multiple-input multiple-output) are explicitly exploited in order to achieve $\mathscr{H}_\infty$ almost synchronization.

To express it clearly, the objective of this paper is to design decentralized observer-based protocols which reduce the $\mathscr{H}_\infty$ norm of the closed-loop transfer function from disturbance to synchronization errors to *any* arbitrarily small value for homogeneous networks of non-introspective, linear, single-input single-output (SISO) agents. We consider directional communication links.

The design is based on the time-scale structure assignment technique [14]. The controller is parameterized by $\varepsilon \in (0,1]$; synchronization is guaranteed in the absence of disturbances if ε is chosen sufficiently small ε. As the controller is continuous in ε, one may adjust ε, online, within a certain range, to obtain the required accuracy of synchronization. Thus, the proposed design turns out to a non-iterative (one-shot) design.

B. Notations

Throughout the paper, matrix A is represented by $A = [a_{ij}]$ where the element (i,j) of A is shown by a_{ij}. Ker$A$ and ImA denote respectively the kernel and the image of A. The Kronecker product of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as $A \otimes B = [a_{ij}B]$. Let $\|A\|$ denote the induced 2-norm. We adopt $x = \text{col}\{x_i\}$ for $i = 1,\cdots,n$ to denote $x = [x_1^{\mathrm{T}},\cdots,x_n^{\mathrm{T}}]^{\mathrm{T}}$ where x_i's are vectors.

The identity matrix of order n is symbolized by I_n. Let $\mathbf{1}_n \in \mathbb{R}^n$ be the vector with all entries equal to one. The real part of a complex number λ is represented by $\mathrm{Re}(\lambda)$. The open left-half and open right-half complex planes are represented

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by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. For a transfer function $T(s)$, the $\mathscr{H}_\infty$ norm is denoted $\|T(s)\|_\infty$. For a space $\mathscr{V}$, the orthogonal complement is shown by $\mathscr{V}^\perp$.

II. Homogeneous Multi-Agent Systems

A homogeneous multi-agent system is referred to a network of single-input single-output agents described by identical linear time-invariant models as

$$\text{Agent } i: \dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{G}\mathbf{w}_i, \qquad \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \tag{1a}$$

in which $i \in \mathfrak{S} \triangleq \{1, \cdots, N\}$. Also, $\mathbf{x}_i \in \mathbb{R}^n$ $(n \geq 1)$ is the state, $\mathbf{u}_i \in \mathbb{R}$ is the control, $\mathbf{y}_i \in \mathbb{R}$ is the output, $\mathbf{w}_i \in \mathbb{R}$: $\lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} \mathbf{w}_i^2(t)\,\mathrm{d}t < \infty$ is the external disturbance. We assume the transfer function is not identically zero.

Agents are non-introspective and no self-measurements are available; in other words, agent i does not have access to its own state $\mathbf{x}_i$ or output $\mathbf{y}_i$. Thus, the only information available to agent i is the network measurement which is transmitted over the network based on a fixed communication topology.

The network's communication topology is described by a directed graph $\mathscr{G}$ whose nodes correspond with agents. If an edge exists from node j to i, a positive real weight a_{ij} is given to the edge. We assume that no self-loops are allowed; i.e. $a_{ii} = 0$. The graph $\mathscr{G}$ is associated with the Laplacian matrix $\mathsf{G} = [g_{ij}]$ where $g_{ij} = -a_{ij}$ for $i, j \in \mathfrak{S}$, $i \neq j$ and $g_{ii} = \sum_{j=1}^N a_{ij}$. It follows that $\lambda = 0$ is an eigenvalue of G with a right eigenvector $\mathbf{1}_N$. The network measurement given to agent $i \in \mathfrak{S}$ is:

$$\zeta_i = \sum_{j=1}^N a_{ij}(\mathbf{y}_i - \mathbf{y}_j) = \sum_{j=1}^N g_{ij}\mathbf{y}_j \tag{1d}$$

In addition, it is assumed that agents are capable of exchanging additional information over the network. The transmission of this additional information conforms with the network's communication topology and facilitates the design of a distributed observer. Thus, agent i has access to the following quantity:

$$\hat{\zeta}_i = \sum_{j=1}^N g_{ij}\eta_j \tag{1e}$$

where $\eta_j \in \mathbb{R}$ depends on the state of the protocol of agent $j \in \mathfrak{S}$; it will be specified later when a dynamic protocol is introduced.

III. $\mathscr{H}_\infty$ Almost Synchronization

This section tackles the problem of $\mathscr{H}_\infty$ almost synchronization for a network of agents described by (1).

A. Problem Formulation

We define the following vectors which are formed by stacking the corresponding vectors of each agent:

$$w \triangleq \operatorname{col}\{\mathbf{w}_i\}, \qquad \zeta \triangleq \operatorname{col}\{\zeta_i\}, \qquad \text{for } i \in \mathfrak{S}$$

Define the mutual disagreement between any pair of agents:

$$\mathfrak{e}_{i,j} \triangleq y_i - y_j, \quad \text{for } i, j \in \mathfrak{S},\ i > j \tag{2}$$

The stacking column vector of all mutual disagreements is denoted $\mathfrak{e}$. Synchronization is achieved if $\mathfrak{e} = 0$. We define the following transfer function with the appropriate dimension:

$$\mathfrak{e} = T_{w\mathfrak{e}}(s)w \tag{3}$$

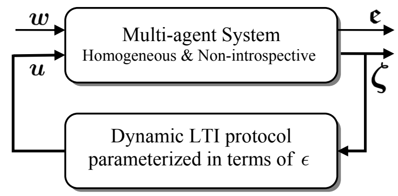

Fig. 1: The block diagram of the closed-loop control system.

Fig. 1 depicts the block diagram of the multi-agent system. The problem that we cope with is precisely defined in Problem 1.

Problem 1: Consider the multi-agent system (1) with a communication topology $\mathscr{G}$. Given a set of network graphs $\mathscr{G}^*$ and any $\gamma > 0$, the "$\mathscr{H}_\infty$ almost synchronization" problem is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathscr{G} \in \mathscr{G}^*$, the closed-loop transfer function from w to $\mathfrak{e}$ satisfies

$$\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma \qquad ◀$$

For agent $i \in \mathfrak{S}$, let the protocol which maps ζ_i and $\hat{\zeta}_i$ to $\mathbf{u}_i$ have the internal state $\xi_i \in \mathbb{R}^{\mathsf{q}_i}$ for some integer $\mathsf{q}_i > 0$, and takes the following general form

$$\begin{cases} \dot{\xi}_i = \mathscr{A}_c(\varepsilon)\,\xi_i + \mathscr{B}_c(\varepsilon)\operatorname{col}\{\zeta_i, \hat{\zeta}_i\} & \text{(4a)} \\ \mathbf{u}_i = \mathscr{C}_c(\varepsilon)\,\xi_i + \mathscr{D}_c(\varepsilon)\operatorname{col}\{\zeta_i, \hat{\zeta}_i\} & \text{(4b)} \end{cases}$$

B. Preliminaries and Assumptions

The conditions under which development of the desired protocol is viable are given in terms of geometric subspaces and an appropriate set of networks. Geometric subspaces and its application to exact disturbance decoupling are explained in [15] and [16]. The application to almost disturbance decoupling is presented in [17].

Let $\mathscr{V}^*_{\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$ be the maximal $(\mathbf{A}+\mathbf{B}\mathbf{F})$-invariant subspace of $\mathbb{R}^n$ contained in $\operatorname{Ker}\mathbf{C}$ such that the eigenvalues of $(\mathbf{A}+\mathbf{B}\mathbf{F})$ belong to $\mathbb{C}^-$ for some $\mathbf{F}$. The supremal $\mathscr{L}_p$-almost controllability subspace 'contained' in $\operatorname{Ker}\mathbf{C}$ is represented by $\mathscr{R}^*_{\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$. Let $\mathscr{S}^*_{\operatorname{Im}\mathbf{B}}(\mathbf{A},\mathbf{B},\mathbf{C})$ denote the minimal $(\mathbf{A}+\mathbf{K}\mathbf{C})$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}\mathbf{B}$ such that the eigenvalues of $(\mathbf{A}+\mathbf{K}\mathbf{C})$ belong to $\mathbb{C}^-$ for some $\mathbf{K}$. Eventually, we define the following subspaces of the state space.

$$\mathscr{V}_{b,\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathscr{V}^*_{\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C}) \oplus \mathscr{R}^*_{\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$$
$$\mathscr{S}_{b,\operatorname{Im}\mathbf{B}}(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathscr{V}_{b,\operatorname{Ker}\mathbf{B}^{\mathrm{T}}}(\mathbf{A}^{\mathrm{T}},\mathbf{C}^{\mathrm{T}},\mathbf{B}^{\mathrm{T}}))^\perp$$

Assumption 1: We make the following assumptions.

1) $(\mathbf{A},\mathbf{B})$ is stabilizable;
2) $(\mathbf{C},\mathbf{A})$ is detectable;
3) $\operatorname{Im}\mathbf{G} \subset \mathscr{V}_{b,\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$;
4) $\mathscr{S}_{b,\operatorname{Im}\mathbf{G}}(\mathbf{A},\mathbf{G},\mathbf{C}) \subset \mathscr{V}_{b,\operatorname{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$;
5) $\mathscr{S}_{b,\operatorname{Im}\mathbf{G}}(\mathbf{A},\mathbf{G},\mathbf{C}) \subset \operatorname{Ker}\mathbf{C}$;

6) The matrix triples $(\mathbf{A},\mathbf{B},\mathbf{C})$ and $(\mathbf{A},\mathbf{G},\mathbf{C})$ have no invariant zeros on the imaginary axis.

The geometric subspaces can be computed by virtue of the special coordinate basis (scb) proposed by [18] (reviewed in Appendix I) using available software, either numerically [19] or symbolically [20].

Definition 1: For given $\beta > 0$ and integer $N_0 \geq 1$, $\mathscr{G}_\beta$ is the set of graphs composed of N nodes where $N \leq N_0$ such that every $\mathscr{G} \in \mathscr{G}_\beta$ has a directed spanning tree, and every nonzero eigenvalue of its Laplacian, denoted λ_i for $i = 1,\cdots,N$, satisfies $\mathrm{Re}\{\lambda_i\} > \beta$.

A directed graph $\mathscr{G}$ has a directed spanning tree if it has a node from which there are directed paths to every other nodes. According to [8], the Laplacian G associated with $\mathscr{G} \in \mathscr{G}_\beta$ has a simple eigenvalue at zero and the rest are located in $\mathbb{C}^+$.

C. Protocol Development

In this section, a family of dynamic protocols is presented, which solves the problem of $\mathscr{H}_\infty$ almost synchronization as stated in Problem 1. More clearly, it is shown that there exists a distributed observer-based protocol parameterized in terms of a tuning parameter $\varepsilon \in (0,1]$ in the form of

$$\begin{cases} \dot{\hat{\mathbf{x}}}_i = (\mathbf{A} - \mathbf{B}\mathbf{F}_{\mathrm{con}}(\varepsilon))\hat{\mathbf{x}}_i + \mathbf{K}_{\mathrm{obs}}(\varepsilon)(\zeta_i - \hat{\zeta}_i) & \text{(5a)} \\ \mathbf{u}_i = -\mathbf{F}_{\mathrm{con}}(\varepsilon)\hat{\mathbf{x}}_i & \text{(5b)} \end{cases}$$

where $\hat{\mathbf{x}}_i \in \mathbb{R}^n$, and $\hat{\zeta}_i$ is given by (1e) where $\eta_j = \mathbf{C}\hat{\mathbf{x}}_j$. We present a step-by-step design procedure for determining the gains $\mathbf{F}_{\mathrm{con}}(\varepsilon)$ and $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$. The algorithm makes use of the special coordinate basis (scb); see Appendix I.

• **Step 1:** Find nonsingular transformations Γ_{x}, Γ_{u} and Γ_{y} in order to represent the SISO system characterized by the matrix triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ into the scb as stated in Appendix I. Therefore, we have

$$\mathbf{x}_i = \Gamma_{\mathrm{x}} x_i, \qquad \mathbf{y}_i = \Gamma_{\mathrm{y}} y_{\mathrm{d},i}, \qquad \mathbf{u}_i = \Gamma_{\mathrm{u}} u_{\mathrm{d},i} \tag{6}$$

where $x_i = \mathrm{col}\{x_{\mathrm{a},i}^-, x_{\mathrm{a},i}^+, x_{\mathrm{d},i}\}$; $x_{\mathrm{a},i}^- \in \mathbb{R}^{n_{\mathrm{a}}^-}$, $x_{\mathrm{a},i}^+ \in \mathbb{R}^{n_{\mathrm{a}}^+}$, $x_{\mathrm{d},i} \in \mathbb{R}^{n_{\mathrm{d}}}$, and $n = n_{\mathrm{a}}^- + n_{\mathrm{a}}^+ + n_{\mathrm{d}}$. We represent $\Gamma_{\mathrm{x}}^{-1}\mathbf{G} = \mathrm{stack}\{G_{\mathrm{a}}^-, G_{\mathrm{a}}^+, G_{\mathrm{d}}\}$. The system dynamics are given by:

$$\dot{x}_{\mathrm{a},i}^- = A_{\mathrm{a}}^- x_{\mathrm{a},i}^- + L_{\mathrm{ad}}^- y_{\mathrm{d},i} + G_{\mathrm{a}}^- \mathbf{w}_i \tag{7a}$$
$$\dot{x}_{\mathrm{a},i}^+ = A_{\mathrm{a}}^+ x_{\mathrm{a},i}^+ + L_{\mathrm{ad}}^+ y_{\mathrm{d},i} + G_{\mathrm{a}}^+ \mathbf{w}_i \tag{7b}$$
$$\dot{x}_{\mathrm{d},i} = A_{\mathrm{d}} x_{\mathrm{d},i} + B_{\mathrm{d}}(u_{\mathrm{d},i} + E_{\mathrm{da}}^- x_{\mathrm{a},i}^- + E_{\mathrm{da}}^+ x_{\mathrm{a},i}^+ + E_{\mathrm{dd}} x_{\mathrm{d},i}) + G_{\mathrm{d}} \mathbf{w}_i \tag{7c}$$
$$y_{\mathrm{d},i} = C_{\mathrm{d}} x_{\mathrm{d},i} \tag{7d}$$

The dimensions of the variables as well as the size and structure of the system matrices conform with the scb stated in Appendix I.

• **Step 2:** Select the feedback gains F_{a}^+ and F_{d} such that the following matrices become Hurwitz stable:

$$A_{\mathrm{d}}^* = A_{\mathrm{d}} - B_{\mathrm{d}} F_{\mathrm{d}}, \qquad A_{\mathrm{ss}} = A_{\mathrm{a}}^+ - L_{\mathrm{ad}}^+ F_{\mathrm{a}}^+$$

Since the pair $(A_{\mathrm{d}}, B_{\mathrm{d}})$ is controllable and the pair $(A_{\mathrm{a}}^+, L_{\mathrm{ad}}^+)$ is stabilizable under Assumption 1-(1), the existence of F_{a}^+ and F_{d} is guaranteed. The dimensions of the gains F_{a}^+ and F_{d} are $1 \times n_{\mathrm{a}}^+$ and $1 \times n_{\mathrm{d}}$, respectively.

• **Step 3:** Consider $\varepsilon \in (0,1]$, and define $\check{S}$ as

$$\check{S}(\varepsilon) = \mathrm{diag}\{1, \varepsilon, \cdots, \varepsilon^{n_{\mathrm{d}}-2}, \varepsilon^{n_{\mathrm{d}}-1}\} \in \mathbb{R}^{n_{\mathrm{d}} \times n_{\mathrm{d}}} \tag{8}$$
$$F_{\mathrm{d}\varepsilon} = \varepsilon^{-n_{\mathrm{d}}} F_{\mathrm{d}} \check{S} \tag{9}$$

where ε is the tuning parameter and will be specified later.

• **Step 4:** Form $\mathscr{F}_{\mathrm{d}} = \mathscr{F}_{\mathrm{dd}} + \mathscr{F}_{\mathrm{d}\varepsilon}$ where $\mathscr{F}_{\mathrm{dd}} = \begin{bmatrix} E_{\mathrm{da}}^- & E_{\mathrm{da}}^+ & E_{\mathrm{dd}} \end{bmatrix}$ and $\mathscr{F}_{\mathrm{d}\varepsilon} = \begin{bmatrix} 0 & F_{\mathrm{d}\varepsilon} C_{\mathrm{d}}^{\mathrm{T}} F_{\mathrm{a}}^+ & F_{\mathrm{d}\varepsilon} \end{bmatrix}$. Find $\mathbf{F}_{\mathrm{con}}(\varepsilon)$:

$$\mathbf{F}_{\mathrm{con}}(\varepsilon) = \Gamma_{\mathrm{u}} \mathscr{F}_{\mathrm{d}} \Gamma_{\mathrm{x}}^{-1} \tag{10}$$

• **Step 5:** Similar to Step 1, find nonsingular transformations $\bar{\Gamma}_{\mathrm{x}}$, $\bar{\Gamma}_{\mathrm{w}}$ and $\bar{\Gamma}_{\mathrm{y}}$ in order to represent the SISO system characterized by the matrix triple $(\mathbf{A},\mathbf{G},\mathbf{C})$ into the scb as stated in Appendix I. For simplicity, we keep the notation used in Step 1 unchanged and place bars on the variables, matrices, and their dimensions. Then choosing

$$\mathbf{x}_i = \bar{\Gamma}_{\mathrm{x}} \bar{x}_i, \qquad \mathbf{y}_i = \bar{\Gamma}_{\mathrm{y}} \bar{y}_{\mathrm{d},i}, \qquad \mathbf{w}_i = \bar{\Gamma}_{\mathrm{w}} w_{\mathrm{d},i} \tag{11}$$

where $\bar{x}_i = \mathrm{col}\{\bar{x}_{\mathrm{a},i}^-, \bar{x}_{\mathrm{a},i}^+, \bar{x}_{\mathrm{d},i}\}$. Let $\bar{\Gamma}_{\mathrm{x}}^{-1}\mathbf{B} = \mathrm{stack}\{\bar{B}_{\mathrm{a}}^-, \bar{B}_{\mathrm{a}}^+, \bar{B}_{\mathrm{d}}\}$. The system is then given by:

$$\dot{\bar{x}}_{\mathrm{a},i}^- = \bar{A}_{\mathrm{a}}^- \bar{x}_{\mathrm{a},i}^- + \bar{L}_{\mathrm{ad}}^- \bar{y}_{\mathrm{d},i} + \bar{B}_{\mathrm{a}}^- \mathbf{u}_i \tag{12a}$$
$$\dot{\bar{x}}_{\mathrm{a},i}^+ = \bar{A}_{\mathrm{a}}^+ \bar{x}_{\mathrm{a},i}^+ + \bar{L}_{\mathrm{ad}}^+ \bar{y}_{\mathrm{d},i} + \bar{B}_{\mathrm{a}}^+ \mathbf{u}_i \tag{12b}$$
$$\dot{\bar{x}}_{\mathrm{d},i} = \bar{A}_{\mathrm{d}} \bar{x}_{\mathrm{d},i} + \bar{G}_{\mathrm{d}}(w_{\mathrm{d},i} + \bar{E}_{\mathrm{da}}^- \bar{x}_{\mathrm{a},i}^- + \bar{E}_{\mathrm{da}}^+ \bar{x}_{\mathrm{a},i}^+ + \bar{E}_{\mathrm{dd}} \bar{x}_{\mathrm{d},i}) + \bar{B}_{\mathrm{d}} \mathbf{u}_i \tag{12c}$$
$$\bar{y}_{\mathrm{d},i} = \bar{C}_{\mathrm{d}} \bar{x}_{\mathrm{d},i} \tag{12d}$$

The dimensions of the variables, the size and structure of the matrices conform with the scb stated in Appendix I.

• **Step 6:** By an appropriate selection of the observer gain $\bar{K}_{\mathrm{a}}^+$, make the following matrix Hurwitz stable:

$$\bar{A}_{\mathrm{ss}} = \bar{A}_{\mathrm{a}}^+ - \bar{K}_{\mathrm{a}}^+ \bar{E}_{\mathrm{da}}^+ \tag{13}$$

Such $\bar{K}_{\mathrm{a}}^+$ exists under Assumption 1-(2). Find $\bar{P}_{\mathrm{d}} = \bar{P}_{\mathrm{d}}^{\mathrm{T}} > 0$ which solves the following algebraic Riccati equation:

$$\bar{A}_{\mathrm{d}} \bar{P}_{\mathrm{d}} + \bar{P}_{\mathrm{d}} \bar{A}_{\mathrm{d}}^{\mathrm{T}} - 2\tau \bar{P}_{\mathrm{d}} \bar{C}_{\mathrm{d}}^{\mathrm{T}} \bar{C}_{\mathrm{d}} \bar{P}_{\mathrm{d}} = -\mathrm{I}_{\bar{n}_{\mathrm{d}}}$$

where $0 < \tau \leq \beta$. The existence of such $\bar{P}_{\mathrm{d}}$ follows from the observability of the pair $(\bar{C}_{\mathrm{d}}, \bar{A}_{\mathrm{d}})$. Now, define $\bar{K}_{\mathrm{d}} = \bar{P}_{\mathrm{d}} \bar{C}_{\mathrm{d}}^{\mathrm{T}}$. We point out that $\bar{K}_{\mathrm{a}}^+ \in \mathbb{R}^{\bar{n}_{\mathrm{a}}^+}$ and $\bar{K}_{\mathrm{d}} \in \mathbb{R}^{\bar{n}_{\mathrm{d}}}$.

• **Step 7:** Define

$$\tilde{S}(\varepsilon) = \mathrm{diag}\{\varepsilon^{-(\bar{n}_{\mathrm{d}}-1)}, \varepsilon^{-(\bar{n}_{\mathrm{d}}-2)}, \cdots, \varepsilon^{-2}, \varepsilon^{-1}, 1\} \tag{14}$$
$$\bar{K}_{\mathrm{d}\varepsilon} = \varepsilon^{-\bar{n}_{\mathrm{d}}} \tilde{S}^{-1} \bar{K}_{\mathrm{d}} \tag{15}$$

• **Step 8:** Form $\bar{\mathscr{K}}_{\mathrm{d}\varepsilon} \in \mathbb{R}^n$ as below:

$$\bar{\mathscr{K}}_{\mathrm{d}\varepsilon} = \begin{bmatrix} 0 \\ \bar{K}_{\mathrm{a}}^+ \bar{G}_{\mathrm{d}}^{\mathrm{T}} \bar{K}_{\mathrm{d}\varepsilon} \\ \bar{K}_{\mathrm{d}\varepsilon} \end{bmatrix} \tag{16}$$

Now, one may obtain $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$ using

$$\mathbf{K}_{\mathrm{obs}}(\varepsilon) = \bar{\Gamma}_{\mathrm{x}} \bar{\mathscr{K}}_{\mathrm{d}\varepsilon} \bar{\Gamma}_{\mathrm{y}}^{-1} \tag{17}$$

Theorem 1 formalizes the result.

Theorem 1: Under Assumption 1 and for the set $\mathscr{G}_\beta$, the parameterized protocol (5), where $\mathbf{F}_{\mathrm{con}}(\varepsilon)$ is selected as in

(10) and $\mathbf{K}_{\rm obs}(\varepsilon)$ is selected as in (17), solves Problem 1. Precisely, the following hold

1) for any given $\beta > 0$, there exists an $\varepsilon_1^* \in (0,1]$ such that for every $\varepsilon \in (0,\varepsilon_1^*]$, synchronization is accomplished in the absence of disturbance; i.e. $\forall \varepsilon \in (0,\varepsilon_1^*]$ when $w = 0$
$$\mathfrak{e}_{i,j} = y_i - y_j \to 0, \quad \forall i,j \in \mathfrak{G}, i > j \quad \text{as} \quad t \to \infty$$
2) for any given $\gamma > 0$, there exists an $\varepsilon_2^* \in (0,\varepsilon_1^*]$ such that for every $\varepsilon \in (0,\varepsilon_2^*]$, the closed-loop transfer function from w to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$.

Proof: See Appendix II. ■

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APPENDIX I
SPECIAL COORDINATE BASIS

In this appendix, we specify the special coordinate basis (scb) proposed by [18] for linear SISO systems, see [21, Theorem 5.2.1]. Consider a SISO system described by

$$\Sigma : \{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{18}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}$ is the control, and $\mathbf{y} \in \mathbb{R}$ is the output. There exist nonsingular state, output and input transformations $\Gamma_{\rm x} \in \mathbb{R}^{n\times n}$, $\Gamma_{\rm y} \in \mathbb{R}$ and $\Gamma_{\rm u} \in \mathbb{R}$ which decompose the state space of Σ into three subspaces $x_{\rm a}^-$, $x_{\rm a}^+$ and $x_{\rm d}$; i.e.

$$\mathbf{x} = \Gamma_{\rm x}\,{\rm col}\{x_{\rm a}^-, x_{\rm a}^+, x_{\rm d}\}, \qquad \mathbf{y} = \Gamma_{\rm y}\, y_{\rm d}, \qquad \mathbf{u} = \Gamma_{\rm u}\, u_{\rm d}$$

where $x_{\rm a}^- \in \mathbb{R}^{n_{\rm a}^-}$, $x_{\rm a}^+ \in \mathbb{R}^{n_{\rm a}^+}$, $x_{\rm d} \in \mathbb{R}^{n_{\rm d}}$, and $n = n_{\rm a}^- + n_{\rm a}^+ + n_{\rm d}$. The system is then described by:

$$\begin{aligned}
\dot{x}_{\rm a}^- &= A_{\rm a}^- x_{\rm a}^- + L_{\rm ad}^- y_{\rm d},\\
\dot{x}_{\rm a}^+ &= A_{\rm a}^+ x_{\rm a}^+ + L_{\rm ad}^+ y_{\rm d}\\
\dot{x}_{\rm d} &= A_{\rm d} x_{\rm d} + B_{\rm d}(u_{\rm d} + E_{\rm da}^- x_{\rm a}^- + E_{\rm da}^+ x_{\rm a}^+ + E_{\rm dd} x_{\rm d})\\
y_{\rm d} &= C_{\rm d} x_{\rm d}
\end{aligned}$$

in which $L_{\rm ad}^-$, $L_{\rm ad}^+$, $E_{\rm da}^-$, $E_{\rm da}^+$, and $E_{\rm dd}$ are some constant matrixes of appropriate dimensions, and

$$A_{\rm d} = \begin{bmatrix} 0 & {\rm I}_{n_{\rm d}-1} \\ 0 & 0 \end{bmatrix}, B_{\rm d} = \begin{bmatrix} 0_{n_{\rm d}-1} \\ 1 \end{bmatrix}, C_{\rm d} = \begin{bmatrix} 1 & 0_{n_{\rm d}-1} \end{bmatrix}$$

$\lambda(A_{\rm a}^-) \in \mathbb{C}^-$ and $\lambda(A_{\rm a}^+) \in \mathbb{C}^+$ are the invariant zeros of the systems, where we have assumed that the system has no invariant zeros on the imaginary axis; $n_{\rm d}$ is the relative degree of the system Σ. For SISO systems, output transformation can be simply chosen to be equal to 1; i.e. $\Gamma_{\rm y} = 1$.

Clearly, $(C_{\rm d}, A_{\rm d})$ form an observable pair. In fact, the system Σ is observable (detectable) if and only if the pair $(C_{\rm obs}, A_{\rm obs})$ is observable (detectable), where

$$C_{\rm obs} = \begin{bmatrix} E_{\rm da}^- & E_{\rm da}^+ \end{bmatrix}, \qquad A_{\rm obs} = \begin{bmatrix} A_{\rm a}^- & 0 \\ 0 & A_{\rm a}^+ \end{bmatrix}$$

Moreover, $(A_{\rm d}, B_{\rm d})$ form a controllable pair. The system Σ is then controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable), where

$$A_{\rm con} = \begin{bmatrix} A_{\rm a}^- & 0 \\ 0 & A_{\rm a}^+ \end{bmatrix} \qquad B_{\rm con} = \begin{bmatrix} L_{\rm ad}^- \\ L_{\rm ad}^+ \end{bmatrix}$$

The geometric subspaces can be expressed in terms of appropriate unions of subspaces that describe the scb of Σ.

Property 1 ([14]): Suppose the state space is described by $x_{\rm a}^- \oplus x_{\rm a}^+ \oplus x_{\rm d}$.

- $x_{\rm a}^- \oplus x_{\rm d}$ spans $\mathscr{V}_{b,\mathrm{Ker}\mathbf{C}}$; • $x_{\rm a}^+$ spans $\mathscr{S}_{b,\mathrm{Im}\mathbf{B}}$.

Appendix II
Proof: Theorem 1

Define the estimation error as $\tilde{\mathbf{x}}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$, and find the dynamics according to (1) and (5). It gives rise to

$$\dot{\tilde{\mathbf{x}}}_i = \mathbf{A}\tilde{\mathbf{x}}_i + \mathbf{G}\mathbf{w}_i - \mathbf{K}_{\rm obs}(\varepsilon)(\zeta_i - \hat{\zeta}_i) \tag{19}$$

where $\zeta_i - \hat{\zeta}_i = \sum_{j=1}^N g_{ij}\mathbf{C}\tilde{\mathbf{x}}_j$ and $\mathbf{K}_{\rm obs}(\varepsilon)$, which is given by (17), is found using the coordinates corresponding to the scb with respect to the triple $(\mathbf{A},\mathbf{G},\mathbf{C})$. Thus, we introduce the following transformations:

$$\tilde{\mathbf{x}}_i = \bar{\Gamma}_{\rm x}\tilde{x}_i, \qquad \mathbf{C}\tilde{\mathbf{x}}_i = \bar{\Gamma}_{\rm y}\tilde{y}_{{\rm d},i}, \qquad \mathbf{w}_i = \bar{\Gamma}_{\rm w} w_{{\rm d},i}$$

where $\tilde{x}_i = \mathrm{col}\{\tilde{x}_{{\rm a},i}^-, \tilde{x}_{{\rm a},i}^+, \tilde{x}_{{\rm d},i}\}$; $\tilde{x}_{{\rm a},i}^- \in \mathbb{R}^{\bar{n}_{\rm a}^-}$, $\tilde{x}_{{\rm a},i}^+ \in \mathbb{R}^{\bar{n}_{\rm a}^+}$, $\tilde{x}_{{\rm d},i} \in \mathbb{R}^{\bar{n}_{\rm d}}$, and $n = \bar{n}_{\rm a}^- + \bar{n}_{\rm a}^+ + \bar{n}_{\rm d}$. We define $\zeta_i - \hat{\zeta}_i = \bar{\Gamma}_{\rm y}\tilde{\zeta}_{{\rm d},i}$ where $\tilde{\zeta}_{{\rm d},i} = \sum_{j=1}^N g_{ij}\tilde{y}_{{\rm d},j}$. Define $\tilde{z}_{{\rm s},i} = \tilde{x}_{{\rm a},i}^+ - \bar{K}_{\rm a}^+\bar{G}_{\rm d}^{\rm T}\tilde{x}_{{\rm d},i}$, and consider the following scalings:

$$\tilde{z}_{{\rm s}\varepsilon,i} = \varepsilon\tilde{z}_{{\rm s},i}, \qquad \tilde{x}_{{\rm d}\varepsilon,i} = \tilde{S}\tilde{x}_{{\rm d},i} \tag{20}$$

Recalling (13), the dynamics of the system are given by:

$$\dot{\tilde{x}}_{{\rm a},i}^- = \bar{A}_{\rm a}^-\tilde{x}_{{\rm a},i}^- + \varepsilon^{\bar{n}_{\rm d}-1}\bar{L}_{\rm ad}^-\bar{C}_{\rm d}\tilde{x}_{{\rm d}\varepsilon,i} \tag{21a}$$

$$\dot{\tilde{z}}_{{\rm s}\varepsilon,i} = \bar{A}_{\rm ss}\tilde{z}_{{\rm s}\varepsilon,i} + \varepsilon\bar{E}_{\rm sa}^-\tilde{x}_{{\rm a},i}^- + \varepsilon\bar{E}_{{\rm sd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i} + \varepsilon\bar{G}_{\rm ss}\mathbf{w}_i \tag{21b}$$

$$\begin{aligned}\varepsilon\dot{\tilde{x}}_{{\rm d}\varepsilon,i} = {}& \bar{A}_{\rm d}\tilde{x}_{{\rm d}\varepsilon,i} + \varepsilon\bar{G}_{\rm d}(\bar{E}_{\rm da}^-\tilde{x}_{{\rm a},i}^- + \bar{E}_{{\rm dd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i}) \\ & + \bar{G}_{\rm d}\bar{E}_{\rm da}^+\tilde{z}_{{\rm s}\varepsilon,i} + \varepsilon\bar{G}_{\rm d}\bar{\Gamma}_{\rm w}^{-1}\mathbf{w}_i - \bar{K}_{\rm d}\tilde{\zeta}_{{\rm d},i}^*\end{aligned} \tag{21c}$$

where $\bar{E}_{{\rm sd}\varepsilon} = (\bar{A}_{\rm ss}\bar{K}_{\rm a}^+\bar{G}_{\rm d}^{\rm T} + \bar{L}_{\rm ad}^+\bar{C}_{\rm d} - \bar{K}_{\rm a}^+\bar{E}_{\rm dd})\tilde{S}^{-1}$, $\bar{G}_{\rm ss} = -\bar{K}_{\rm a}^+\bar{\Gamma}_{\rm w}^{-1}$, $\bar{E}_{\rm sa}^- = -\bar{K}_{\rm a}^+\bar{E}_{\rm da}^-$, $\bar{E}_{{\rm dd}\varepsilon} = (\bar{E}_{\rm dd} + \bar{E}_{\rm da}^+\bar{K}_{\rm a}^+\bar{G}_{\rm d}^{\rm T})\tilde{S}^{-1}$, and $\tilde{\zeta}_{{\rm d},i}^* = \sum_{k=1}^N g_{ik}\bar{C}_{\rm d}\tilde{x}_{{\rm d}\varepsilon,k}$. It is easy to verify that $\|\varepsilon\bar{E}_{{\rm sd}\varepsilon}\| = \mathscr{O}(\varepsilon)$, $\|\varepsilon\bar{E}_{\rm sa}^-\| = \mathscr{O}(\varepsilon)$, $\|\varepsilon\bar{G}_{\rm ss}\| = \mathscr{O}(\varepsilon)$, $\|\varepsilon\bar{G}_{\rm d}\bar{E}_{\rm da}^-\| = \mathscr{O}(\varepsilon)$, $\|\varepsilon\bar{G}_{\rm d}\bar{\Gamma}_{\rm w}^{-1}\| = \mathscr{O}(\varepsilon)$, and $\|\varepsilon\bar{E}_{{\rm dd}\varepsilon}\| = \mathscr{O}(\varepsilon)$.

The control law is $\mathbf{u}_i = -\mathbf{F}_{\rm con}(\varepsilon)\hat{\mathbf{x}}_i = -\mathbf{F}_{\rm con}(\varepsilon)(\mathbf{x}_i - \tilde{\mathbf{x}}_i)$, where $\mathbf{F}_{\rm con}(\varepsilon)$ is selected as (10). In Step 1, using (6), we found (7). Thus, all we need to do is to express $u_{{\rm d},i}$ in terms of the coordinates corresponding to the scb with respect to the triple $(\mathbf{A},\mathbf{B},\mathbf{C})$. Representing $\tilde{\mathbf{x}}_i$ in that coordinates, one obtains $\tilde{\mathbf{x}}_i = \Gamma_{\rm x}\check{\tilde{x}}_i$ where $\check{\tilde{x}}_i = \mathrm{col}\{\check{\tilde{x}}_{{\rm a},i}^-, \check{\tilde{x}}_{{\rm a},i}^+, \check{\tilde{x}}_{{\rm d},i}\}$. It implies that $\check{\tilde{x}}_i = \Gamma_{\rm x}^{-1}\bar{\Gamma}_{\rm x}\tilde{x}_i$. According to Property 1, one can show

- $\check{\tilde{x}}_{{\rm a},i}^- \oplus \check{\tilde{x}}_{{\rm d},i}$ spans $\mathscr{V}_{b,\mathrm{Ker}\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$;
- $\tilde{x}_{{\rm a},i}^+$ spans $\mathscr{S}_{b,\mathrm{Im}\mathbf{G}}(\mathbf{A},\mathbf{G},\mathbf{C})$.

Therefore, according to Assumptions 1-(4),(5), $\check{\tilde{x}}_{{\rm a},i}^+ \subset (\tilde{x}_{{\rm a},i}^- \oplus \tilde{x}_{{\rm d},i})$ and $(\mathrm{Ker}\,\mathbf{C})^\perp \subset (\tilde{x}_{{\rm a},i}^- \oplus \tilde{x}_{{\rm d},i})$. Represent $\check{\tilde{x}}_{{\rm d},i} = \mathrm{col}\{\check{\tilde{x}}_{j{\rm d},i}\}$ for $j = 1,\cdots,n_{\rm d}$; thus, $C_{\rm d}\check{\tilde{x}}_{{\rm d},i} = \check{\tilde{x}}_{1{\rm d},i} \in (\mathrm{Ker}\,\mathbf{C})^\perp$. Therefore, one concludes:

- Assumptions 1-(4),(5) imply that $\check{\tilde{x}}_{1{\rm d},i}$ and $\check{\tilde{x}}_{{\rm a},i}^+$ are expressed in terms of $\tilde{x}_{{\rm a},i}^-$ and $\tilde{x}_{{\rm d},i}$.
- Assumption 1-(3) implies that $G_{\rm a}^+ = 0$ since $\mathrm{Im}\,\mathbf{G}$ is expressed in terms of $x_{{\rm a},i}^-$ and $x_{{\rm d},i}$.

Define: $s_{{\rm d},i} = F_{\rm a}^+x_{{\rm a},i}^+ + y_{{\rm d},i}$ and $z_{{\rm d},i} = C_{\rm d}^{\rm T}F_{\rm a}^+x_{{\rm a},i}^+ + x_{{\rm d},i}$ which implies that $s_{{\rm d},i} = C_{\rm d}z_{{\rm d},i}$. According to the state-feedback gain (10), $u_{{\rm d},i} = -\mathscr{F}_{\rm d}(x_i - \check{\tilde{x}}_i)$, which is written as $u_{{\rm d},i} = -\mathscr{F}_{\rm dd}x_i - F_{{\rm d}\varepsilon}z_{{\rm d},i} + \widetilde{u}_{1,i} + \widetilde{u}_{2,i}$ where $\widetilde{u}_{1,i} = \mathscr{F}_{\rm dd}\check{\tilde{x}}_i$ and $\widetilde{u}_{2,i} = F_{{\rm d}\varepsilon}C_{\rm d}^{\rm T}F_{\rm a}^+\check{\tilde{x}}_{{\rm a},i}^+ + F_{{\rm d}\varepsilon}\check{\tilde{x}}_{{\rm d},i}$. Partition $F_{\rm d} = [f_1,\cdots,f_{n_{\rm d}}]$ where $f_k \in \mathbb{R}$ for $k = 1,\cdots,n_{\rm d}$. Let $F_{{\rm d}\varepsilon}\check{\tilde{x}}_{{\rm d},i} = \varepsilon^{-n_{\rm d}}(f_1\check{\tilde{x}}_{1{\rm d},i} + \varepsilon\sum_{k=2}^{n_{\rm d}}\varepsilon^{k-2}f_k\check{\tilde{x}}_{k{\rm d},i})$. It is confirmed that there exist $M_{\rm da}^-$, $M_{\rm ds}$, $M_{{\rm dd}\varepsilon}$, $N_{\rm da}^-$, $N_{{\rm dd}\varepsilon}$, $\check{M}_{\rm da}^-$, $\check{M}_{\rm ds}$, and $\check{M}_{{\rm dd}\varepsilon}$ (which are uniformly bounded in ε) that

$$B_{\rm d}\widetilde{u}_{1,i} = M_{\rm da}^-\tilde{x}_{{\rm a},i}^- + \varepsilon^{-1}M_{\rm ds}\tilde{z}_{{\rm s}\varepsilon,i} + M_{{\rm dd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i}$$

$$B_{\rm d}\widetilde{u}_{2,i} = \varepsilon^{-n_{\rm d}}(N_{\rm da}^-\tilde{x}_{{\rm a},i}^- + N_{{\rm dd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i} + \varepsilon\check{M}_{\rm da}^-\tilde{x}_{{\rm a},i}^- + \check{M}_{\rm ds}\tilde{z}_{{\rm s}\varepsilon,i} + \varepsilon\check{M}_{{\rm dd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i})$$

Introducing $x_{{\rm a}\varepsilon,i}^- = \varepsilon x_{{\rm a},i}^-$ and $z_{{\rm d}\varepsilon,i} = \check{S}z_{{\rm d},i}$, the closed-loop equations are described by:

$$\begin{aligned}\dot{x}_{{\rm a}\varepsilon,i}^- &= A_{\rm a}^-x_{{\rm a}\varepsilon,i}^- - \varepsilon L_{\rm ad}^-F_{\rm a}^+x_{{\rm a},i}^+ + \varepsilon L_{\rm ad}^-C_{\rm d}z_{{\rm d}\varepsilon,i} + \varepsilon G_{\rm a}^-\mathbf{w}_i \\ \dot{x}_{{\rm a},i}^+ &= A_{\rm ss}x_{{\rm a},i}^+ + L_{\rm ad}^+C_{\rm d}z_{{\rm d}\varepsilon,i} \\ \varepsilon\dot{z}_{{\rm d}\varepsilon,i} &= A_{\rm d}^*z_{{\rm d}\varepsilon,i} + \varepsilon\check{S}L_{\rm dd}^*C_{\rm d}z_{{\rm d}\varepsilon,i} + \varepsilon\check{S}L_{\rm da}^+x_{{\rm a},i}^+ + \varepsilon\check{S}G_{\rm d}\mathbf{w}_i \\ &\quad + N_{\rm da}^-\tilde{x}_{{\rm a},i}^- + N_{{\rm dd}\varepsilon}\tilde{x}_{{\rm d}\varepsilon,i} + H_{\rm ds}\tilde{z}_{{\rm s}\varepsilon,i} + \varepsilon H_{\rm da}^-\tilde{x}_{{\rm a},i}^- + \varepsilon H_{\rm dd}\tilde{x}_{{\rm d}\varepsilon,i}\end{aligned}$$

where $L_{\rm dd}^* = C_{\rm d}^{\rm T}F_{\rm a}^+L_{\rm ad}^+$ and $L_{\rm da}^+ = C_{\rm d}^{\rm T}F_{\rm a}^+A_{\rm ss}$. Also, $H_{\rm ds} = \check{M}_{\rm ds} + \varepsilon^{n_{\rm d}-1}M_{\rm ds}$, $H_{\rm da}^- = \check{M}_{\rm da}^- + \varepsilon^{n_{\rm d}-1}M_{\rm da}^-$, and $H_{\rm dd} = \check{M}_{{\rm dd}\varepsilon} + \varepsilon^{n_{\rm d}-1}M_{{\rm dd}\varepsilon}$. It can be confirmed that the norms of $\varepsilon\check{S}L_{\rm dd}^*$, $\varepsilon\check{S}L_{\rm da}^+$, $\varepsilon\check{S}G_{\rm d}$, $\varepsilon H_{\rm da}^-$, and $\varepsilon H_{\rm dd}$ are of order ε. Define

$$\begin{aligned}\tilde{\mathbf{z}}_i &= \mathrm{col}\{\tilde{x}_{{\rm a},i}^-, \tilde{x}_{{\rm d}\varepsilon,i}, \tilde{z}_{{\rm s}\varepsilon,i}\}, & \mathbf{z}_i &= \mathrm{col}\{x_{{\rm a},i}^+, x_{{\rm a}\varepsilon,i}^-, z_{{\rm d}\varepsilon,i}\} \\ \bar{\mathscr{A}} &= \mathrm{diag}\{\bar{A}_{\rm a}^-, \bar{A}_{\rm d}, \bar{A}_{\rm ss}\}, & \mathscr{A} &= \mathrm{diag}\{A_{\rm ss}, A_{\rm a}^-, A_{\rm d}^*\} \\ \bar{\mathscr{S}} &= \mathrm{diag}\{\mathrm{I}_{\bar{n}_{\rm a}^-}, \varepsilon\mathrm{I}_{\bar{n}_{\rm d}}, \mathrm{I}_{\bar{n}_{\rm a}^+}\}, & \mathscr{S} &= \mathrm{diag}\{\mathrm{I}_{n_{\rm a}^+}, \mathrm{I}_{n_{\rm a}^-}, \varepsilon\mathrm{I}_{n_{\rm d}}\}\end{aligned}$$

The closed-loop equations are given in the compact form by:

$$\mathscr{S}\dot{\mathbf{z}}_i = \mathscr{A}\mathbf{z}_i + \mathscr{L}\mathbf{z}_i + \varepsilon\mathscr{E}\mathbf{w}_i + \mathscr{D}\tilde{\mathbf{z}}_i \tag{22a}$$

$$\bar{\mathscr{S}}\dot{\tilde{\mathbf{z}}}_i = \bar{\mathscr{A}}\tilde{\mathbf{z}}_i + \bar{\mathscr{L}}\tilde{\mathbf{z}}_i + \varepsilon\bar{\mathscr{E}}\mathbf{w}_i - \textstyle\sum_{k=1}^N g_{ik}\bar{\mathscr{D}}\tilde{\mathbf{z}}_k \tag{22b}$$

where $\mathscr{L} = \varepsilon\mathscr{L}_\varepsilon + \mathscr{L}_0$, $\|\varepsilon\mathscr{L}_\varepsilon\| = \mathscr{O}(\varepsilon)$ and $\bar{\mathscr{L}} = \varepsilon\bar{\mathscr{L}}_\varepsilon + \bar{\mathscr{L}}_0$, $\|\varepsilon\bar{\mathscr{L}}_\varepsilon\| = \mathscr{O}(\varepsilon)$; $\mathscr{D} = \varepsilon\mathscr{D}_\varepsilon + \mathscr{D}_0$ in which $\|\varepsilon\mathscr{D}_\varepsilon\| = \mathscr{O}(\varepsilon)$; the norms of $\mathscr{L}_0$, $\bar{\mathscr{L}}_0$ and $\mathscr{D}_0$ are uniformly bounded.

$$\mathscr{L}_\varepsilon = \begin{bmatrix} 0 & 0 & 0 \\ -L_{\rm ad}^-F_{\rm a}^+ & 0 & L_{\rm ad}^-C_{\rm d} \\ \check{S}L_{\rm da}^+ & 0 & \check{S}L_{\rm dd}^*C_{\rm d}\end{bmatrix}, \quad \mathscr{L}_0 = \begin{bmatrix} 0 & 0 & L_{\rm ad}^+C_{\rm d} \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

$$\bar{\mathscr{L}}_\varepsilon = \begin{bmatrix} 0 & 0 & 0 \\ \bar{G}_{\rm d}\bar{E}_{\rm da}^- & \bar{G}_{\rm d}\bar{E}_{{\rm dd}\varepsilon} & 0 \\ \bar{E}_{\rm sa}^- & \bar{E}_{{\rm sd}\varepsilon} & 0\end{bmatrix}, \quad \bar{\mathscr{E}} = \begin{bmatrix} 0 \\ \bar{G}_{\rm d}\bar{\Gamma}_{\rm w}^{-1} \\ \bar{G}_{\rm ss}\end{bmatrix}$$

$$\bar{\mathscr{L}}_0 = \begin{bmatrix} 0 & \varepsilon^{\bar{n}_{\rm d}-1}\bar{L}_{\rm ad}^-\bar{C}_{\rm d} & 0 \\ 0 & 0 & \bar{G}_{\rm d}\bar{E}_{\rm da}^+ \\ 0 & 0 & 0\end{bmatrix}, \quad \mathscr{E} = \begin{bmatrix} 0 \\ G_{\rm a}^- \\ \check{S}G_{\rm d}\end{bmatrix}$$

$$\mathscr{D}_\varepsilon = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ H_{\rm da}^- & H_{\rm dd} & 0\end{bmatrix}, \quad \bar{\mathscr{D}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{K}_{\rm d}\bar{C}_{\rm d} & 0 \\ 0 & 0 & 0\end{bmatrix}$$

$$\mathscr{D}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ N_{\rm da}^- & N_{{\rm dd}\varepsilon} & H_{\rm ds}\end{bmatrix}$$

For $i \in \mathfrak{S}$, collect the states as $\chi = \mathrm{col}\{\mathbf{z}_i\}$ and $\tilde{\chi} = \mathrm{col}\{\tilde{\mathbf{z}}_i\}$. Then, we can write

$$(\mathrm{I}_N \otimes \mathscr{S})\dot{\chi} = (\mathrm{I}_N \otimes \mathscr{A})\chi + (\mathrm{I}_N \otimes \mathscr{L})\chi + \varepsilon(\mathrm{I}_N \otimes \mathscr{E})w + (\mathrm{I}_N \otimes \mathscr{D})\tilde{\chi} \tag{23a}$$

$$(\mathrm{I}_N \otimes \bar{\mathscr{S}})\dot{\tilde{\chi}} = \left((\mathrm{I}_N \otimes \bar{\mathscr{A}}) - (\mathsf{G} \otimes \bar{\mathscr{D}})\right)\tilde{\chi} + (\mathrm{I}_N \otimes \bar{\mathscr{L}})\tilde{\chi} + \varepsilon(\mathrm{I}_N \otimes \bar{\mathscr{E}})w \tag{23b}$$

Let $\mathbf{1}, \mathbf{1}_L \in \mathbb{R}^N$: $\mathsf{G}\mathbf{1} = 0$ and $\mathbf{1}_L^{\mathrm{T}}\mathsf{G} = 0$. Suppose the Jordan form of G is obtained using the matrix U which is chosen as $U = [\bar{U},\, \mathbf{1}] \Rightarrow (U^{-1})^{\mathrm{T}} = [\bar{U}_L,\, \mathbf{1}_L]$. Thus, one can find $U^{-1}\mathsf{G}U = \mathrm{diag}\{\Delta, 0\}$. It implies that $\mathsf{G}U = [\check{\mathsf{G}},\, 0]$ where $\check{\mathsf{G}} = \bar{U}\Delta$. We introduce the following state transformations

$$\begin{bmatrix} e \\ e_0 \end{bmatrix} = (U^{-1} \otimes \mathrm{I}_n)\chi, \qquad \begin{bmatrix} \tilde{e} \\ \tilde{e}_0 \end{bmatrix} = (U^{-1} \otimes \mathrm{I}_n)\tilde{\chi} \tag{24}$$

where $e_0, \tilde{e}_0 \in \mathbb{R}^n$. Let $\bar{N} = N - 1$. Then, we find the following set of equations.

$$(\mathrm{I}_{\bar{N}} \otimes \mathscr{S})\dot{e} = (\mathrm{I}_{\bar{N}} \otimes (\mathscr{A} + \mathscr{L}_0))\,e + \varepsilon(\mathrm{I}_{\bar{N}} \otimes \mathscr{L}_\varepsilon)e + (\mathrm{I}_{\bar{N}} \otimes \mathscr{D})\tilde{e} + \varepsilon\,(\bar{U}_L^{\mathrm{T}} \otimes \mathscr{E})w \tag{25a}$$

$$(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{S}})\dot{\tilde{e}} = \big(\mathrm{I}_{\bar{N}} \otimes (\bar{\mathscr{A}} + \bar{\mathscr{L}}_0) - \Delta \otimes \bar{\mathscr{D}}\big)\,\tilde{e} + \varepsilon(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{L}}_\varepsilon)\tilde{e} + \varepsilon\,(\bar{U}_L^{\mathrm{T}} \otimes \bar{\mathscr{E}})w \tag{25b}$$

$\zeta = (\check{\mathsf{G}} \otimes \Gamma_{\mathrm{y}} C_{\mathrm{d}} \Gamma_{\mathrm{x}}^*)e$ for some Γ_{x}^*. Choose $\rho > 0$ such that $\zeta^{\mathrm{T}}\zeta \le \rho^2 e^{\mathrm{T}}e$. The matrix $\mathscr{A}$ is Hurwitz stable because A_{a}^-, A_{ss}, and A_{d}^* are Hurwitz stable. Due to the upper block-triangular structure of $\mathscr{L}_0$ where the diagonal are zero, the matrix $(\mathrm{I}_{\bar{N}} \otimes (\mathscr{A} + \mathscr{L}_0))$ is upper block-triangular and Hurwitz stable. It implies that there exists a symmetric $\mathscr{Q} > (\rho^2 + 4)\mathrm{I}_n$ such that the Lyapunov equation:

$$(\mathscr{A} + \mathscr{L}_0)^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\mathscr{A} + \mathscr{L}_0) = -\mathscr{Q}$$

has a solution $\mathscr{P} = \mathscr{P}^{\mathrm{T}} > 0$ which is block-diagonal with the block sizes that correspond to the block sizes in $\mathscr{S}$. It guarantees that $\mathscr{P}$ and $\mathscr{S}$ commute. Let $V_c = e^{\mathrm{T}}(\mathrm{I}_{\bar{N}} \otimes \mathscr{S}\mathscr{P})e$. Taking derivative gives rise to

$$\dot{V}_c \le -\zeta^{\mathrm{T}}\zeta - 3\|e\|^2 - (1 - 2\varepsilon\rho)\|e\|^2 + 2\mu_{\mathrm{d}}\|e\|\|\tilde{e}\| + 2\varepsilon\rho_{\mathrm{w}}\|e\|\|w\|$$

where $\rho_{\mathrm{w}} = \|(\bar{U}_L^{\mathrm{T}} \otimes \mathscr{P}\mathscr{E})\|$

$$\rho \ge \max_{\varepsilon \in (0,1]} \|\mathscr{P}\mathscr{L}_\varepsilon\|, \qquad \mu_{\mathrm{d}} \ge \max_{\varepsilon \in (0,1]} \|\mathscr{P}\mathscr{D}\|,$$

Since ρ is bounded, there exists sufficiently small $\varepsilon_{11} \in (0,1]$ such that $1 - 2\varepsilon\rho > 0$ for every $\varepsilon \in (0, \varepsilon_{11}]$; thus,

$$\dot{V}_c \le -\zeta^{\mathrm{T}}\zeta - 3\|e\|^2 + 2\mu_{\mathrm{d}}\|e\|\|\tilde{e}\| + 2\varepsilon\rho_{\mathrm{w}}\|e\|\|w\|$$

In (25b), let $\bar{\mathscr{A}}^* = \mathrm{I}_{\bar{N}} \otimes (\bar{\mathscr{A}} + \bar{\mathscr{L}}_0) - \Delta \otimes \bar{\mathscr{D}}$. From the structure of Δ, it is observed that $\bar{\mathscr{A}}^*$ is upper block-triangular. Thus, $\bar{\mathscr{A}}^*$ is Hurwitz stable if and only if all matrices on the main diagonal are Hurwitz stable. In other words, $\bar{\mathscr{A}} + \bar{\mathscr{L}}_0 - \lambda\bar{\mathscr{D}}$ must be Hurwitz stable for all λ's which are nonzero eigenvalues of the Laplacian G. Since $\bar{\mathscr{A}} - \lambda\bar{\mathscr{D}}$ is a block-diagonal matrix and $\bar{\mathscr{L}}_0$ is upper block-triangular with zero diagonal, the eigenvalues of $\bar{\mathscr{A}}^*$ are determined by the eigenvalues of $\bar{A}_{\mathrm{a}}^-$, $\bar{A}_{\mathrm{ss}}$, and $\bar{A}_{\mathrm{d}}^* = \bar{A}_{\mathrm{d}} - \lambda\bar{K}_{\mathrm{d}}\bar{C}_{\mathrm{d}}$. The matrix $\bar{A}_{\mathrm{a}}^-$ is Hurwitz stable by definition, and $\bar{A}_{\mathrm{ss}}$ was made Hurwitz stable in Step 6. It can be confirmed that $\bar{A}_{\mathrm{d}}^*$ is Hurwitz stable. To see that we recall $\bar{K}_{\mathrm{d}} = \bar{P}_{\mathrm{d}}\bar{C}_{\mathrm{d}}^{\mathrm{T}}$ and show

$$\begin{aligned}(\bar{A}_{\mathrm{d}}^*)\bar{P}_{\mathrm{d}} + \bar{P}_{\mathrm{d}}(\bar{A}_{\mathrm{d}}^*)^{\mathrm{H}} &= \bar{A}_{\mathrm{d}}\bar{P}_{\mathrm{d}} + \bar{P}_{\mathrm{d}}\bar{A}_{\mathrm{d}}^{\mathrm{T}} - 2\,\mathrm{Re}(\lambda)\bar{P}_{\mathrm{d}}\bar{C}_{\mathrm{d}}^{\mathrm{T}}\bar{C}_{\mathrm{d}}\bar{P}_{\mathrm{d}} \\ &= \bar{A}_{\mathrm{d}}\bar{P}_{\mathrm{d}} + \bar{P}_{\mathrm{d}}\bar{A}_{\mathrm{d}}^{\mathrm{T}} - 2\tau\bar{P}_{\mathrm{d}}\bar{C}_{\mathrm{d}}^{\mathrm{T}}\bar{C}_{\mathrm{d}}\bar{P}_{\mathrm{d}} \\ &\quad - 2(\mathrm{Re}(\lambda) - \tau)\bar{P}_{\mathrm{d}}\bar{C}_{\mathrm{d}}^{\mathrm{T}}\bar{C}_{\mathrm{d}}\bar{P}_{\mathrm{d}} \le -\mathrm{I}_{\bar{n}_{\mathrm{b}}}\end{aligned}$$

Notice since $\mathscr{G} \in \mathscr{G}_\beta$, $\mathrm{Re}(\lambda) > \beta \ge \tau > 0$ if $\lambda \ne 0$. It follows that $\bar{A}_{\mathrm{d}}^*$ is Hurwitz stable. Hence, $\bar{\mathscr{A}} + \bar{\mathscr{L}}_0 - \lambda\bar{\mathscr{D}}$ and accordingly $\bar{\mathscr{A}}^*$ is Hurwitz stable for $\beta \ge \tau$. Let $\lambda_N = 0$ and $\lambda_i \in \mathbb{C}^+$ for $i \in \{1, \cdots, N-1\}$. It is easy to show that, for every $i \in \{1, \cdots, N-1\}$, there exists a $\tilde{\mathrm{Q}}_i = \tilde{\mathrm{Q}}_i^{\mathrm{T}} > 0$ such that the Lyapunov equation

$$(\bar{\mathscr{A}} + \bar{\mathscr{L}}_0 - \lambda_i\bar{\mathscr{D}})^{\mathrm{H}}\tilde{\mathrm{P}}_i + \tilde{\mathrm{P}}_i(\bar{\mathscr{A}} + \bar{\mathscr{L}}_0 - \lambda_i\bar{\mathscr{D}}) = -\tilde{\mathrm{Q}}_i$$

has a unique solution $\tilde{\mathrm{P}}_i = \tilde{\mathrm{P}}_i^{\mathrm{T}} > 0$ which is block-diagonal with the block sizes that correspond to the block sizes in $\bar{\mathscr{S}}$. Let $\tilde{\mathrm{q}}_i > 0$ be such that $\tilde{\mathrm{q}}_i\mathrm{I} \le \tilde{\mathrm{Q}}_i$, and let $\tilde{\eta}_i = \|\tilde{\mathrm{P}}_i\bar{\mathscr{D}}\|$. The block-diagonal matrix $\tilde{\mathscr{P}}$ which is constructed as

$$\tilde{\mathscr{P}} = \mathrm{diag}\{\delta_1\tilde{\mathrm{P}}_1, \cdots, \delta_{N-1}\tilde{\mathrm{P}}_{N-1}\} \tag{26}$$

where $\delta_{N-1} = 1$, $\delta_i = \delta_{i+1}\dfrac{\tilde{\mathrm{q}}_i\tilde{\mathrm{q}}_{i+1}}{9\tilde{\eta}_i^2}$ for $i = 1, \cdots, N-2$ (which implies $\|\tilde{\mathscr{P}}\|$ is bounded for any $\beta > 0$) solves the Lyapunov function

$$(\bar{\mathscr{A}}^*)^{\mathrm{H}}\tilde{\mathscr{P}} + \tilde{\mathscr{P}}\bar{\mathscr{A}}^* = -\tilde{\mathscr{Q}} \tag{27}$$

for a symmetric $\tilde{\mathscr{Q}} > (3 + \mu_{\mathrm{d}}^2)\mathrm{I}_{\bar{N}n}$. $\tilde{\mathscr{P}}$ commutes with $(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{S}})$. Choose

$$V_o = \tilde{e}^{\mathrm{T}}(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{S}})\tilde{\mathscr{P}}\tilde{e}$$

taking derivative yields

$$\dot{V}_o \le -(2 + \mu_{\mathrm{d}}^2)\|\tilde{e}\|^2 - (1 - 2\varepsilon\tilde{\rho})\|\tilde{e}\|^2 + 2\varepsilon\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|\|w\|$$

where

$$\tilde{\rho} \ge \max_{\varepsilon \in (0,1]} \|\tilde{\mathscr{P}}(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{L}}_\varepsilon)\|, \qquad \tilde{\rho}_{\mathrm{w}} \ge \|\tilde{\mathscr{P}}(\bar{U}_L^{\mathrm{T}} \otimes \bar{\mathscr{E}})\|$$

Due to boundedness of $\tilde{\mathscr{P}}$ for any $\beta > 0$, $\tilde{\rho}$ and $\tilde{\rho}_{\mathrm{w}}$ are bounded for any $\beta > 0$. As a consequence, one may find an $\varepsilon_{22} \in (0,1]$ such that $1 - 2\varepsilon\tilde{\rho} > 0$ for every $\varepsilon \in (0, \varepsilon_{22}]$; thus, for every $\varepsilon \in (0, \varepsilon_{22}]$, we obtain

$$\dot{V}_o \le -(2 + \mu_{\mathrm{d}}^2)\|\tilde{e}\|^2 + 2\varepsilon\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|\|w\|$$

Choose $V = V_c + V_o$. Let $\varepsilon_1^* = \min\{\varepsilon_{11}, \varepsilon_{22}\}$. For every $\varepsilon \in (0, \varepsilon_1^*]$, an upper bound on $\dot{V}$ is given by

$$\dot{V} \le -\zeta^{\mathrm{T}}\zeta - \|e\|^2 - \|\tilde{e}\|^2 + (\varepsilon\sigma_{\mathrm{w}})^2\|w\|^2$$

where $\sigma_{\mathrm{w}} \ge \sqrt{2}\max\{\rho_{\mathrm{w}}, \tilde{\rho}_{\mathrm{w}}\}$. Hence, the origin is globally exponentially stable in the absence of disturbance. It follows from Kalman-Yakubovich-Popov Lemm that in the presence of disturbances, $\|T_{w\zeta}\|_\infty \le \varepsilon\sigma_{\mathrm{w}}$. Put alternatively, the impact of w on ζ can be made arbitrarily small in the sense of the $\mathscr{H}_\infty$ norm of the transfer function.

To reach the objective, it is required to show that the influence of disturbance on every mutual disagreement, $\mathfrak{e}_{i,j}$, can be made small to any arbitrary degree. Define $\mathfrak{e}_{i,j} = T_{w\mathfrak{e}}^{i,j}(s)w$.

Then, from [13, Lemma 3], it follows that if $\|T_{w\zeta}\|_\infty \le \varepsilon\sigma_{\mathrm{w}}$, there exists $\hat{\sigma} > 0$ such that $\|T_{w\mathfrak{e}}^{i,j}\|_\infty < \varepsilon\hat{\sigma}$. Therefore, for any given $\gamma > 0$, there exists an $\varepsilon_2^* \in (0, \varepsilon_1^*]$ such that every $\varepsilon \in (0, \varepsilon_2^*]$ yields $\|T_{w\mathfrak{e}}^{i,j}\|_\infty < \gamma$. Thus, any desired accuracy of synchronization can be obtained by reducing ε in the range $(0, \varepsilon_1^*]$.