Formation is possible when agents exchange output errors of formation; i.e. $y_{f,i}$'s. Therefore, the network information (1e) is to be modified to

$$\begin{aligned}\zeta_{f,i} &= \zeta_i - \sum_{j=1}^{N} l_{ij} f_j \\ &= \sum_{j=1}^{N} l_{ij} y_{f,j} \end{aligned} \tag{20}$$

The formation controller relies extensively on the fact that shaping is viable to any invertible system of uniform rank $n_q \geq n_{q0}$ which has no invariant zeros. Thus, Lemma 2 guarantees existence of local feedback laws to shape the system into the desired structure as (5).

In view of a multi-agent system (5), we propose the parameterized dynamic protocol

$$\dot{\hat{x}}_i = A\hat{x}_i + B(u_i + R\hat{x}_i) - \epsilon^{-1}K(\zeta_{f,i} - \hat{\zeta}_i) \tag{21a}$$
$$u_i = \epsilon^{-n_q} FS\hat{x}_i - R_1 f_i \tag{21b}$$

for $i \in \mathcal{S}$. In (21), $\hat{x}_i \in \mathbb{R}^{pn_q}$ and $\epsilon \in (0, 1]$ is the tuning parameter. Matrix S is as (8) and K is partitioned as (9). F, $\bar{K}$ and K_1 are chosen similar to the procedure presented in Subsection 4.3. The quantity $\zeta_{f,i}$ is found using (20). Considering $\eta_i = C\hat{x}_i$, $\hat{\zeta}_i$ is given by (1e). Theorem 3 states the result formally.

Theorem 3 *Under Assumption 1 and for the set $\mathcal{G}_\beta$, the problem of $\mathcal{H}_\infty$ almost formation is solvable; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter $\epsilon \in (0, 1]$, of the form*

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\epsilon)\chi_i + \mathcal{B}_i(\epsilon) \text{ col}\{\zeta_{f,i}, \hat{\zeta}_i, y_{m,i}\} & \text{(22a)} \\ \bar{u}_i = \mathcal{C}_i(\epsilon)\chi_i + \mathcal{D}_i(\epsilon) \text{ col}\{\zeta_{f,i}, \hat{\zeta}_i, y_{m,i}\} + \mathcal{F}_i(f_i) & \text{(22b)} \end{cases}$$

where $\chi_i \in \mathbb{R}^{q_i}$ and $i \in \mathcal{S}$ such that

(i) for any $\beta > 0$, there exists an $\epsilon_1^* \in (0, 1]$ such that if $\epsilon \in (0, \epsilon_1^*]$, the desired formation is attained in the absence of disturbance; i.e. $\forall \epsilon \in (0, \epsilon_1^*]$ when $\boldsymbol{w} = 0$

$$\bar{\mathfrak{e}}_{i,j} \to 0 \quad as \quad t \to \infty$$

(ii) for any given $\gamma > 0$, there exists an $\epsilon_2^* \in (0, \epsilon_1^*]$ such that if $\epsilon \in (0, \epsilon_2^*]$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}_f$ satisfies

$$\|T_{w\mathfrak{e}_f}(s)\|_\infty < \gamma$$

PROOF. See Appendix C.

Remark 3 *It is worth noting that the given protocol imposes no restrictions on the agreement trajectories; that is, although agents establish the desired configuration, it is not clear where the whole system heads to. The formation control can be combined with the regulation problem, arising the problem of "$\mathcal{H}_\infty$ almost formation with regulation of output consensus" which is illustrated by simulation.*

7 Illustrative Example

The result is illustrated for a network consisting of $N = 4$ right-invertible agents with $p = 1$. The interconnection topology of the network is given by the digraph displayed in Fig. 4a. The corresponding Laplacian is

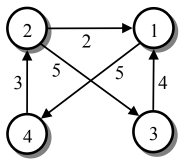
$$L = \begin{bmatrix} 6 & -2 & -4 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & -5 & 5 & 0 \\ -5 & 0 & 0 & 5 \end{bmatrix}$$

The models of agents are given in Appendix E. Disturbances are chosen $w_1 = \sin(t)$, $w_2 = 1$, $w_4 = \sin(2t)$, and $\|w_3\| \leq 5$ which is a uniform random number. The order of the infinite zeros of agent 1 to 4 are respectively 3, 2, 1, and 2. Thus, agent 1 has the largest order of infinite zeros; i.e. $n_{q0} = 3$; we choose $n_q = n_{q0}$.

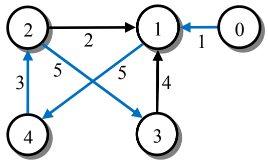
7.1 Network Shaping

The first step is to design a local output feedback for each agent to have an almost identical representation for the network. Agents 1 and 2 are invertible, but agents 3 and 4 need to be squared down. The pre-compensator $\Sigma_{1,3}$ squares down agent 3 and locates the additional invariant zeros at $\{-2, -2, -2, -2, -1\}$. Placing the additional infinite zeros at $\{-2, -2, -1\}$, the interconnection of $\Sigma_{1,4}$ and agent 4 is invertible. The dynamic equations of the compensators are given in Appendix F.1.

Together with the pre-compensators, all the agents are invertible and single-input single-output. Now, we make all of them of the same relative degree n_q. Agents 2, 3, and 4 are compensated by $\Sigma_{2,2}$, $\Sigma_{2,3}$ and $\Sigma_{2,4}$, represented in Appendix F.2.



(a) Primary Network



(b) Augmented Network

Fig. 4. The communication topology of the network.

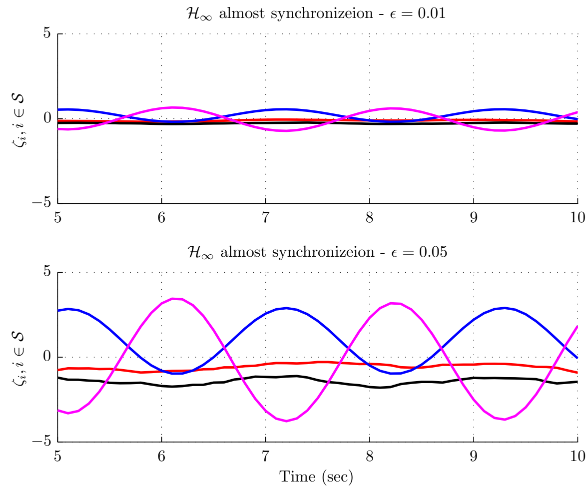


Fig. 5. $\mathcal{H}_\infty$ almost output synchronization. A blow-up of the simulation results: $\epsilon = 0.01$ in the upper plot; $\epsilon = 0.05$ in the lower plot.

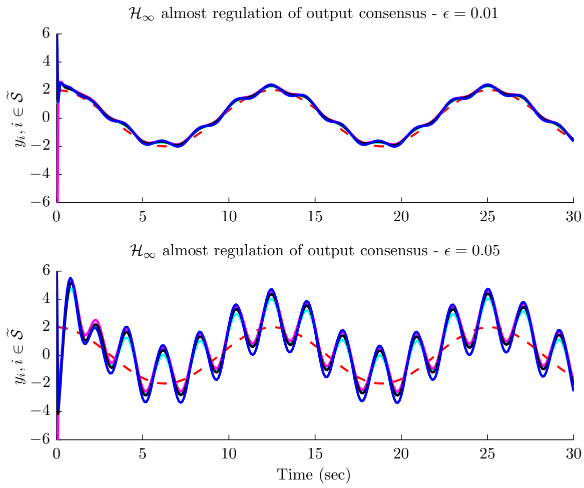


Fig. 6. $\mathcal{H}_\infty$ almost regulation of output consensus for two values of ϵ: the upper plot is for $\epsilon = 0.01$; the lower plot is for $\epsilon = 0.05$. The red dash lines show y_0.

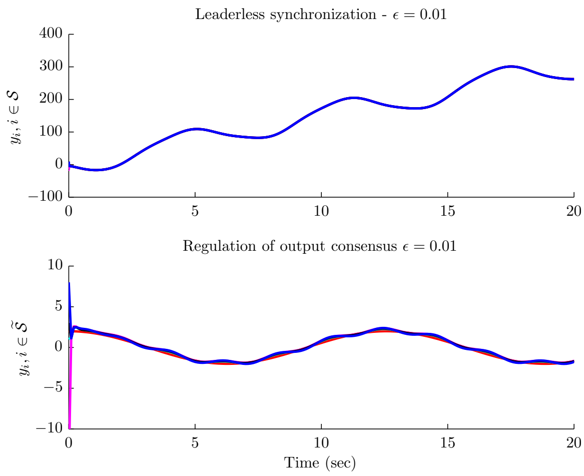


Fig. 7. A comparison between leaderless synchronization (upper) and regulation of output consensus (lower) is given to distinguish the difference in the consensus trajectories. The consensus trajectory in the leaderless case may be unbounded.

After pre-compensation agent 1-4 are of orders 3, 3, 8 and 7, respectively. The third step to achieve an almost identical representation is to design output feedbacks for the pre-compensated agents, which requires linear observers. The required information is presented in Appendix F.3.

7.2 $\mathcal{H}_\infty$ Almost Synchronization

The appropriately shaped agents are now considered and the control law is produced according to (7). We choose $M = 1$ and $R = [0,\ -1,\ 0]$ to have $\lambda(A+BR) = \{0, \pm i\}$ for the synchronization problem. We select $K_1 = -1$. F and $\bar{K}$ are selected such that $\lambda(A+BF) = \{-3, -4, -5\}$ and $\lambda(\widetilde{A}_z) = \{-2, -3\}$. Fig. 5 shows the result for $\epsilon = 0.01$ and $\epsilon = 0.05$. The smaller the value of ϵ is, the smaller ζ_i is, the more $\boldsymbol{w}$ is rejected from the mutual disagreements.

7.3 $\mathcal{H}_\infty$ Almost Regulation of Output Consensus

The consensus trajectory for the multi-agent system is desired to be $\sin(\omega_0 t)$. The exo-system (12) is then given by

$$\bar{A}_0 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \qquad \bar{C}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{23}$$

with $\bar{x}_0(0) = [2,\ 0]^{\mathrm{T}}$. Thus, $n_q = n_{q0} = 3$ is preserved, but the exo-system has to be shaped as desired. If $\bar{B}_0 = [0,\ 1]^{\mathrm{T}}$ is chosen, the resulting system is invertible and one may design a rank-equalizing pre-compensator to change the relative degree to n_q. Transforming the system using the observability matrix renders the exo-system as (18) with

$$R_0 = \begin{bmatrix} 0, & -\omega_0^2, & 0 \end{bmatrix}, \qquad M_0 = 1$$

and $x_0 = [2,\ 0,\ -2\omega_0^2]^{\mathrm{T}}$. Agent 1 is the root of one spanning tree in the network graph, and we link agent 0 with agent 1 with weight $\psi_1 = 1$ as depicted in Fig. 4b. In fact, we choose $\pi = \{1\}$. Let $\omega_0 = 0.5$. Fig. 6 displays the results, where the outputs of all agents are plotted, for $\epsilon = 0.01$ and $\epsilon = 0.05$. As expected, smaller ϵ leads to a better disturbance rejection. To discern the contrast between the leaderless synchronization and the regulation of output consensus, the consensus trajectories for each case are plotted in Fig. 7. It is established from Fig. 7 that the consensus trajectory for the leaderless synchronization is unbounded although $\lambda(A+BR) = \{0, \pm i\}$ implies that the response of the system would be oscillatory; however, the network is driven by disturbance. By contrast, the consensus trajectory in the regulation of output consensus is close to what is desired.

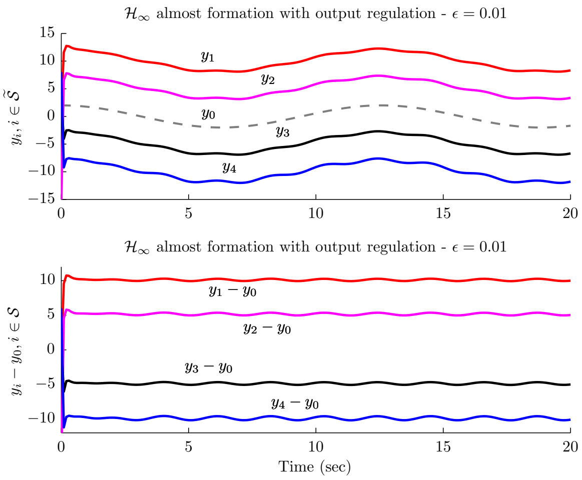


Fig. 8. $\mathcal{H}_\infty$ almost formation control with output regulation; the upper plot shows y_i, $i \in \widetilde{\mathcal{S}}$ and the lower plot depicts the regulation error $y_i - y_0$, $i \in \mathcal{S}$.

7.4 $\mathcal{H}_\infty$ Almost Formation Control with Regulation

This part builds up $\mathcal{H}_\infty$ almost regulation of output consensus into formation control. To be specific, agents have to form a desired configuration while they are forced to follow the reference y_0. This can be portrayed as virtual reference formation control. Simulation is carried out for the same network as before. The formation set is selected as

$$\mathcal{S}_f = \{10, 5, -5, -10\}$$

Thus, the objective is to have

$$y_1 - 10 = y_2 - 5 = y_3 + 5 = y_4 + 10 = y_0 \doteq \sin(0.5t)$$

as time approaches to infinity. The result is given in Fig. 8.

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A Proof of Lemma 1

Before embarking on the proof, we would like to stress that the problem of $\mathcal{H}_\infty$ almost synchronization can be stated equivalently by choosing $\boldsymbol{\zeta}$ as the controlled output and designing a protocol such that $\|T_{w\zeta}(s)\|_\infty < \gamma'$ for some $\gamma' > 0$ where

$$\boldsymbol{\zeta} = T_{w\zeta}(s)\boldsymbol{w}$$

In this case, the objective is to approximately decouple $\boldsymbol{\zeta}$ from $\boldsymbol{w}$ to the desired level in the sense of the $\mathcal{H}_\infty$ norm of $T_{w\zeta}$. We shall show in Lemma 3 that these two formulations are equivalent.

To prove Lemma 1, we first write the closed-loop equations after suitable state transformations; second, the unobservable modes are removed and the order of the system from $\boldsymbol{w}$ to $\boldsymbol{\zeta}$ is reduced. Then, the $\mathcal{H}_\infty$ norm of the transfer function based on the reduced order system is established. Finally, we show that the $\mathcal{H}_\infty$ norm of the transfer function from $\boldsymbol{w}$ to $\mathfrak{e}_{i,j}$ can be made arbitrarily small.

A.1 Closed-loop Equations

Let $x_i^* \triangleq x_i - \hat{x}_i$ be the observation error for agent i; then, in view of (5) and (7), the closed-loop equations for agent i can be written as

$$\dot{x}_i = Ax_i + BRx_i + \epsilon^{-n_q}BFS(x_i - x_i^*) + E_i w_i$$
$$\dot{x}_i^* = Ax_i^* + BRx_i^* + \epsilon^{-1}KC\sum_{j=1}^{N} l_{ij}x_j^* + E_i w_i$$

Let $R = [R_1,\ \bar{R}]$ where $R_1 \in \mathbb{R}^{p\times p}$ and

$$\bar{R} = \begin{bmatrix} R_2 & \cdots & R_{n_q} \end{bmatrix}, \qquad \text{where } R_i \in \mathbb{R}^{p\times p}$$

Consider the following state transformations

$$e_i = Sx_i, \quad z_i = \bar{S}x_i^*, \quad \bar{S} \triangleq \begin{bmatrix} \mathcal{I}_p & 0 \\ -\epsilon\bar{K} & \epsilon\,\mathcal{I}_{p(n_q-1)} \end{bmatrix}$$

The closed-loop equations can be then recast as:

$$\dot{e}_i = \epsilon^{-1}(A + BF)e_i + R_e e_i - \epsilon^{-1}BFS\bar{S}^{-1}z_i + SE_i w_i$$
$$\dot{z}_i = A_z z_i + R_z z_i + \epsilon^{-1}C^{\mathrm{T}}K_1C\sum_{j=1}^{N} l_{ij}z_j + \bar{S}E_i w_i$$

in which

$$R_e = B\begin{bmatrix} \epsilon^{n_q-1}R_1, & \epsilon^{n_q-2}R_2, & \cdots, & R_{n_q} \end{bmatrix}$$
$$R_z = \epsilon B\begin{bmatrix} R_1 + \bar{R}\bar{K}, & \epsilon^{-1}\bar{R} \end{bmatrix}, \quad A_z = \begin{bmatrix} C_1\bar{K} & \epsilon^{-1}C_1 \\ \epsilon\bar{K}'\bar{K} & \bar{K}' \end{bmatrix}$$

where $\bar{K}' = A_1 - \bar{K}C_1$. We split z_i into $z_{1,i} = Cz_i \in \mathbb{R}^p$ and $z_{2,i}$ such that $z_i = \mathrm{col}\{z_{1,i}, z_{2,i}\}$. Let

$$F = \begin{bmatrix} F_1 & F_2 & \cdots & F_{n_q} \end{bmatrix}$$
$$\bar{K} = \begin{bmatrix} \bar{K}_1^{\mathrm{T}} & \bar{K}_2^{\mathrm{T}} & \cdots & \bar{K}_{n_q-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

where $F_i \in \mathbb{R}^{p\times p}$, $\bar{K}_i \in \mathbb{R}^{p\times p}$. We represent

$$FS\bar{S}^{-1} = \begin{bmatrix} F_1^* & F_2^* \end{bmatrix}$$
$$F_1^* = F_1 + \sum_{s=1}^{n_q-1} \epsilon^s F_{s+1}\bar{K}_s$$
$$F_2^* = \begin{bmatrix} F_2 & \epsilon F_3 & \cdots & \epsilon^{n_q-3}F_{n_q-1} & \epsilon^{n_q-2}F_{n_q} \end{bmatrix}$$

Defining $\widetilde{E}_{2,i} \triangleq E_{2,i} - \bar{K}E_{1,i}$ and $\widetilde{E}_z \triangleq \widetilde{A}_z\bar{K} + B_1R_1$, one may show the closed-loop equations as:

$$\epsilon\dot{e}_i = (A+BF)e_i + \epsilon R_e e_i - BF_1^* z_{1,i} - BF_2^* z_{2,i} + \epsilon SE_i w_i$$
$$\epsilon\dot{z}_{1,i} = \epsilon C_1\bar{K}z_{1,i} + C_1 z_{2,i} + K_1 \sum_{j=1}^{N} l_{ij}z_{1,j} + \epsilon E_{1,i}w_i$$
$$\dot{z}_{2,i} = \epsilon\widetilde{E}_z z_{1,i} + \widetilde{A}_z z_{2,i} + \epsilon\widetilde{E}_{2,i}w_i$$

For $i \in \mathcal{S}$, consider the following notations

$$\mathbb{G} = \mathrm{diag}\{SE_i\}, \quad \widetilde{\mathbb{G}}_1 = \mathrm{diag}\{E_{1,i}\}, \quad \widetilde{\mathbb{G}}_2 = \mathrm{diag}\{\widetilde{E}_{2,i}\}$$
$$e = \mathrm{col}\{e_i\}, \quad z_1 = \mathrm{col}\{z_{1,i}\}, \quad z_2 = \mathrm{col}\{z_{2,i}\}$$

Considering the network Laplacian $L = [l_{ij}]$ for $i, j \in \mathcal{S}$, the closed-loop equations for the network are given by

$$\epsilon\dot{e} = (\mathcal{I}_N \otimes (A+BF))\,e + (\mathcal{I}_N \otimes \epsilon R_e)e - (\mathcal{I}_N \otimes BF_1^*)z_1 - (\mathcal{I}_N \otimes BF_2^*)z_2 + \epsilon\mathbb{G}\boldsymbol{w} \quad \text{(A.2a)}$$
$$\epsilon\dot{z}_1 = (\mathcal{I}_N \otimes \epsilon C_1\bar{K} + L \otimes K_1)z_1 + (\mathcal{I}_N \otimes C_1)z_2 + \epsilon\widetilde{\mathbb{G}}_1\boldsymbol{w} \quad \text{(A.2b)}$$
$$\dot{z}_2 = (\mathcal{I}_N \otimes \epsilon\widetilde{E}_z)z_1 + (\mathcal{I}_N \otimes \widetilde{A}_z)z_2 + \epsilon\widetilde{\mathbb{G}}_2\boldsymbol{w} \quad \text{(A.2c)}$$
$$\boldsymbol{\zeta} = (L \otimes C)e \quad \text{(A.2d)}$$

A.2 Reduced-order Dynamics

One right eigenvector of L is $\mathbf{1} \in \mathbb{R}^N$. Let $\mathbf{1}_L$ represent its left eigenvector. Suppose the Jordan form of L is obtained using the matrix U which is chosen as

$$U = \begin{bmatrix} \bar{U} & \mathbf{1} \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} \bar{U}_L^{\mathrm{T}} \\ \mathbf{1}_L^{\mathrm{T}} \end{bmatrix}$$

It implies that $\bar{U}_L^{\mathrm{T}}\bar{U} = \mathcal{I}_{N-1}$, $\bar{U}_L^{\mathrm{T}}\mathbf{1} = \mathbf{1}_L^{\mathrm{T}}\bar{U} = 0$, and $\mathbf{1}_L^{\mathrm{T}}\mathbf{1} = 1$. Thus, one can find

$$U^{-1}LU = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \qquad LU = \begin{bmatrix} \check{L} & 0 \end{bmatrix} \quad \text{(A.3)}$$

where $\check{L} = \bar{U}\Delta$. Since $\mathcal{L} \in \mathcal{G}_\beta$, all eigenvalues of Δ lies in $\mathbb{C}^+$. The following state transformations are introduced:

$$\begin{bmatrix} \bar{e} \\ e^* \end{bmatrix} = (U^{-1} \otimes \mathcal{I}_{pn_q})e \quad \text{(A.4a)}$$
$$\begin{bmatrix} \bar{z}_1 \\ z_1^* \end{bmatrix} = (U^{-1} \otimes \mathcal{I}_p)z_1 \quad \text{(A.4b)}$$
$$\begin{bmatrix} \bar{z}_2 \\ z_2^* \end{bmatrix} = (U^{-1} \otimes \mathcal{I}_{\bar{p}})z_2 \quad \text{(A.4c)}$$

where $\bar{p} = p(n_q - 1)$. Denoting $\bar{N} = N-1$, the states $\bar{e}$, $\bar{z}_1$ and $\bar{z}_2$ are vectors of the dimensions $\bar{N} \times pn_q$, $\bar{N} \times p$ and $\bar{N} \times \bar{p}$, respectively. Obviously, the states e^*, z_1^* and z_2^* are vectors of the dimensions pn_q, p and $\bar{p}$, respectively. Consider the following notations:

$$\widehat{\mathbb{G}}_e = (\bar{U}_L^{\mathrm{T}} \otimes \mathcal{I}_{pn_q})\mathbb{G}, \qquad \mathbb{G}_e^* = (\mathbf{1}_L^{\mathrm{T}} \otimes \mathcal{I}_{pn_q})\mathbb{G}$$
$$\widehat{\mathbb{G}}_{z1} = (\bar{U}_L^{\mathrm{T}} \otimes \mathcal{I}_p)\widetilde{\mathbb{G}}_1, \qquad \mathbb{G}_{z1}^* = (\mathbf{1}_L^{\mathrm{T}} \otimes \mathcal{I}_{pn_q})\widetilde{\mathbb{G}}_1$$
$$\widehat{\mathbb{G}}_{z2} = (\bar{U}_L^{\mathrm{T}} \otimes \mathcal{I}_{\bar{p}})\widetilde{\mathbb{G}}_2, \qquad \mathbb{G}_{z2}^* = (\mathbf{1}_L^{\mathrm{T}} \otimes \mathcal{I}_{\bar{p}})\widetilde{\mathbb{G}}_2$$

As a result, the system dynamics is divided into two subsystems. One subsystem is of order $2pn_q\bar{N}$, and is given by

$$\epsilon\dot{\bar{e}} = (\mathcal{I}_{\bar{N}} \otimes (A+BF))\bar{e} + (\mathcal{I}_{\bar{N}} \otimes \epsilon R_e)\bar{e} - (\mathcal{I}_{\bar{N}} \otimes BF_1^*)\bar{z}_1 - (\mathcal{I}_{\bar{N}} \otimes BF_2^*)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_e\boldsymbol{w} \quad \text{(A.5a)}$$
$$\epsilon\dot{\bar{z}}_1 = (\mathcal{I}_{\bar{N}} \otimes \epsilon C_1\bar{K} + \Delta \otimes K_1)\bar{z}_1 + (\mathcal{I}_{\bar{N}} \otimes C_1)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_{z1}\boldsymbol{w} \quad \text{(A.5b)}$$
$$\dot{\bar{z}}_2 = (\mathcal{I}_{\bar{N}} \otimes \epsilon\widetilde{E}_z)\bar{z}_1 + (\mathcal{I}_{\bar{N}} \otimes \widetilde{A}_z)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_{z2}\boldsymbol{w} \quad \text{(A.5c)}$$
$$\bar{\boldsymbol{\zeta}} = \boldsymbol{\zeta} = (\check{L} \otimes C)\bar{e} \quad \text{(A.5d)}$$

It follows from (A.5d) that $\boldsymbol{\zeta}$ is only affected by $\bar{e}$ since the chosen output for the system only captures disagreement between agents and it depends only on $\mathfrak{e}_{i,j} = y_i - y_j$. The other subsystem is of order $2pn_q$ and is given by

$$\epsilon\dot{e}^* = (A+BF)e^* + \epsilon R_e e^* - BF_1^* z_1^* - BF_2^* z_2^* + \epsilon\,\mathbb{G}_e^* w \quad \text{(A.6a)}$$
$$\epsilon\dot{z}_1^* = \epsilon C_1\bar{K}z_1^* + C_1 z_2^* + \epsilon\,\mathbb{G}_{z1}^* w \quad \text{(A.6b)}$$
$$\dot{z}_2^* = \epsilon\widetilde{E}_z z_1^* + \widetilde{A}_z z_2^* + \epsilon\,\mathbb{G}_{z2}^* w \quad \text{(A.6c)}$$

When all agents have reached an agreement, $\boldsymbol{\zeta}$ is zero which does not impose any constraints on the dynamic system (A.6). In fact, (A.6) determines the consensus trajectories when $\boldsymbol{\zeta} = 0$. It enunciates the fact that the consensus trajectories may be unbounded.

It is then inferred that the objective in the network synchronization is to design a protocol such that the reduced-order dynamics (A.5) vanishes in time. Eventually, the $\mathcal{H}_\infty$ almost synchronization problem for the network of agents is converted to the $\mathcal{H}_\infty$ control of the reduced-order dynamics.

A.3 $\mathcal{H}_\infty$ Analysis

We prove the theorem for $\|T_{w\zeta}\|_\infty$ as the transfer function of the reduced order system (A.5). According to (A.5d), $\|\boldsymbol{\zeta}\| = \rho_\zeta\|\bar{e}\|$ for some ρ_ζ independent of ϵ. As $A+BF$ is Hurwitz stable, there exists $P_c = P_c^{\mathrm{T}} > 0$ which solves the following Lyapunov function:

$$(\mathcal{I}_{\bar{N}} \otimes (A+BF))^{\mathrm{T}} P_c + P_c\,(\mathcal{I}_{\bar{N}} \otimes (A+BF)) = -2\mathcal{I}$$

Choose the positive definite function $W_c = \epsilon \bar{e}^{\mathrm{T}} P_c \bar{e}$ and differentiate it along the trajectories of (A.5a). Denote

$$s_0 = \|P_c(\mathcal{I}_{\bar{N}} \otimes R_e)\| \qquad \rho_1 = \|P_c \widehat{\mathbb{G}}_e\|$$
$$s_1 = \|P_c(\mathcal{I}_{\bar{N}} \otimes BF_1^*)\| \qquad s_2 = \|P_c(\mathcal{I}_{\bar{N}} \otimes BF_2^*)\|$$

Notice $\epsilon s_0 = \mathcal{O}(\epsilon)$ and s_1 and s_2 are bounded. For sufficiently small ϵ, we obtain

$$(1 - \epsilon s_0) > \frac{1}{2} \tag{A.7}$$

Then, one can find the upper bound for $\dot{W}_c$ as

$$\dot{W}_c \le -\|\bar{e}\|^2 + 2\rho_2 \sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\bar{e}\| + 2\epsilon\rho_1\|\bar{e}\|\|\boldsymbol{w}\| \tag{A.8}$$

where $\rho_2 = \sqrt{2}\max\{s_1, s_2\}$. To find the upper bound (A.8), we have made use of the fact that $x + y \le \sqrt{2}\sqrt{x^2 + y^2}$. Since $K_1 < 0$ and $\lambda(\Delta) \in \mathbb{C}^+$, $\Delta \otimes K_1$ is Hurwitz stable. Thus, the Lyapunov equation

$$(\Delta \otimes K_1)^{\mathrm{H}} P_1 + P_1(\Delta \otimes K_1) = -Q_1 \tag{A.9}$$

has a unique solution $P_1 > 0$. Recalling Definition 1, as $\mathcal{L} \in \mathcal{G}_\beta$, $\mathrm{Re}\{\lambda_i(\Delta)\} > \beta$, for $i = 1, \cdots, \bar{N}$.

Proposition 1 *For any $\beta > 0$, there exists a bounded $P_1 > 0$ such that the Lyapunov equation* (A.9) *holds and*

$$\|Q_1\| > 4\mathfrak{q} \tag{A.10}$$

where $\mathfrak{q} = \mathfrak{q}(\beta)$.

PROOF. See Appendix D.

Proposition 1 states that for the set $\mathcal{G}_\beta$, we can construct a block diagonal matrix $P_1 > 0$, which is bounded; P_1 solves the Lyapunov function (A.9) for $Q_1 > 0$ such that $\|Q_1\|$ is bounded from below by a function of $\beta > 0$. Choose $W_1 = \mathfrak{q}\epsilon \bar{z}_1^{\mathrm{T}} P_1 \bar{z}_1$ and differentiate it in time. The upper bound for $\dot{W}_1$ is then given by

$$\begin{aligned}\dot{W}_1 \le -2\mathfrak{q}^2\|\bar{z}_1\|^2 - 2\mathfrak{q}\|\bar{z}_1\|^2(\mathfrak{q} - \epsilon s_3) \\ + 2\mathfrak{q}s_4\|\bar{z}_1\|\|\bar{z}_2\| + 2\mathfrak{q}\epsilon\rho_3\|\bar{z}_1\|\|\boldsymbol{w}\|\end{aligned}$$

where $\rho_3 = \|P_1 \widehat{\mathbb{G}}_{z1}\|$ and

$$s_3 = \|P_1(\mathcal{I}_{\bar{N}} \otimes C_1\bar{K})\| \qquad s_4 = \|P_1(\mathcal{I}_{\bar{N}} \otimes C_1)\|$$

Since $\widetilde{A}_z$ is Hurwitz stable, the equation

$$(\mathcal{I}_{\bar{N}} \otimes \widetilde{A}_z)^{\mathrm{T}} P_2 + P_2(\mathcal{I}_{\bar{N}} \otimes \widetilde{A}_z) = -(2\mathfrak{q} + \mathfrak{q}^{-1}s_4^2)\mathcal{I}$$

has the unique solution $P_2 = P_2^{\mathrm{T}} > 0$. The derivative of $W_2 = \mathfrak{q}\,\bar{z}_2^{\mathrm{T}} P_2 \bar{z}_2$ along the trajectories of (A.5c) is bounded by

$$\dot{W}_2 \le -(2\mathfrak{q}^2 + s_4^2)\|\bar{z}_2\|^2 + 2\mathfrak{q}\epsilon s_5\|\bar{z}_2\|\|\bar{z}_1\| + 2\mathfrak{q}\epsilon\rho_4\|\bar{z}_2\|\|\boldsymbol{w}\|$$

in which

$$s_5 = \|P_2(\mathcal{I}_{\bar{N}} \otimes \widetilde{E}_z)\| \qquad \rho_4 = \|P_2 \widehat{\mathbb{G}}_{z2}\|$$

Consider $W_o = W_1 + W_2$ and differentiate in time. One may find an upper bound for $\dot{W}_o$ as

$$\begin{aligned}\dot{W}_o \le &-\mathfrak{q}^2\|\bar{z}_1\|^2 + 2\mathfrak{q}s_4\|\bar{z}_1\|\|\bar{z}_2\| - s_4^2\|\bar{z}_2\|^2 \\ &- 2\mathfrak{q}\|\bar{z}_1\|^2(\mathfrak{q} - \epsilon s_3) + 2\mathfrak{q}\epsilon s_5\|\bar{z}_2\|\|\bar{z}_1\| - \mathfrak{q}^2\|\bar{z}_2\|^2 \\ &- \mathfrak{q}^2\|\bar{z}_1\|^2 - \mathfrak{q}^2\|\bar{z}_2\|^2 + 2\mathfrak{q}\epsilon\rho_3\|\bar{z}_1\|\|\boldsymbol{w}\| + 2\mathfrak{q}\epsilon\rho_4\|\bar{z}_2\|\|\boldsymbol{w}\|\end{aligned}$$

The first line is a square and is negative. The second line can be made negative by choosing ϵ sufficiently small to have (A.11) satisfied.

$$\mathfrak{q} - \epsilon s_3 > 0 \tag{A.11a}$$
$$2\mathfrak{q}^4 - 2\mathfrak{q}^3\epsilon s_3 - \mathfrak{q}^2\epsilon^2 s_5^2 > 0 \tag{A.11b}$$

Then, it turns out that

$$\dot{W}_o \le -\mathfrak{q}^2\|\bar{z}_1\|^2 - \mathfrak{q}^2\|\bar{z}_2\|^2 + 2\mathfrak{q}\epsilon\rho_5\sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\boldsymbol{w}\|$$

where $\rho_5 = \sqrt{2}\max\{\rho_3, \rho_4\}$. Consequently, we select the following Lyapunov function for the system (A.5):

$$V = (2 + \rho_\zeta^2)W_c + (1 + (2 + \rho_\zeta^2)^2\rho_2^2\,\mathfrak{q}^{-2})W_o \tag{A.12}$$

Differentiating V with respect to time yields

$$\begin{aligned}\dot{V} \le &-\|\bar{e}\|^2 + 2(2 + \rho_\zeta^2)\rho_2\sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\bar{e}\| \\ &- (2 + \rho_\zeta^2)^2\rho_2^2(\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2) \\ &- \rho_\zeta^2\|\bar{e}\|^2 - (\mathfrak{q}^2\|\bar{z}_1\|^2 + \mathfrak{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2) \\ &+ 2\epsilon\rho_6\sqrt{\mathfrak{q}^2\|\bar{z}_1\|^2 + \mathfrak{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2}\|\boldsymbol{w}\|\end{aligned}$$

where $\rho_6 = \sqrt{2}\max\{\rho_1(2 + \rho_\zeta^2), \rho_5(1 + (2 + \rho_\zeta^2)^2\rho_2^2\,\mathfrak{q}^{-2})\}$. The first two lines comprise a square, which is negative. Then, we arrive at

$$\begin{aligned}\dot{V} \le &-\rho_\zeta^2\|\bar{e}\|^2 - (\mathfrak{q}^2\|\bar{z}_1\|^2 + \mathfrak{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2) \\ &+ 2\epsilon\rho_6\sqrt{\mathfrak{q}^2\|\bar{z}_1\|^2 + \mathfrak{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2}\|\boldsymbol{w}\|\end{aligned} \tag{A.13}$$

Completing the square, it gives rise to

$$\dot{V} + \|\boldsymbol{\zeta}\|^2 - (\epsilon\rho_6)^2\|\boldsymbol{w}\|^2 \le 0 \tag{A.14}$$

Therefore, it follows from Kalman-Yakubovich-Popov Lemma (Zhou and Doyle, 1998) that

$$\|T_{w\zeta}\|_\infty < \epsilon\rho_6$$

and the contribution of $\boldsymbol{w}$ to $\boldsymbol{\zeta}$ vanishes as $\epsilon \to 0$. Note that (A.14) is obtained if $\epsilon \in (0, \epsilon_1^*]$ where ϵ_1^* is the largest ϵ which satisfies the conditions (A.7) and (A.11).

We would like to draw attention to the fact that P_1 is found for a set of networks, say $\mathcal{G}_\beta$, not for a given network $\mathcal{L} \in \mathcal{G}_\beta$; accordingly, s_3, s_4, and s_5 are independent of one specific choice for the network graph.

Inequality (A.13) accentuates that $\bar{e}, \bar{z}_1, \bar{z}_2 \to 0$ exponentially fast and agreement is reached if $\boldsymbol{w} = 0$ although it makes no conclusions about the agreement states (e^*, z_1^*, z_2^*). Thus, the agreement trajectories can be nonzero or even unbounded. ■

So far, we have shown that the proposed family of protocols can reject $\boldsymbol{w}$ from $\boldsymbol{\zeta}$ to the desired level. Lemma 3 demonstrates that (7) has a similar decoupling effect on $\mathfrak{e}$. We define

$$\mathfrak{e}_{i,j} = T_{w\mathfrak{e}}^{i,j}(s)\boldsymbol{w}$$

Lemma 3 *Given $\epsilon > 0$ and for γ, $\gamma' > 0$, the following statements are equivalent.*

(1) $\|T_{w\zeta}\|_\infty < \epsilon\gamma$
(2) $\|T_{w\mathfrak{e}}^{i,j}\|_\infty < \epsilon\gamma'$

PROOF. If (2) is given, (1) is deduced for some $\gamma > 0$ since $\boldsymbol{\zeta}$ is the weighted sum of $\mathfrak{e}_{i,j}$'s. To show the other direction, by an appropriate choice of $\mathbb{A}$ (which is Hurwitz stable) and $\mathbb{B}$, the input-output representation of (A.5) is described by

$$\boldsymbol{\zeta} = \epsilon\,(L \otimes C)(s\mathcal{I} - \mathbb{A})^{-1}\mathbb{B}\,\boldsymbol{w} \tag{A.15}$$

We pick one agent arbitrarily. Let it be agent N. Due to zero row-sum property of the Laplacian, we have $\sum_{j=1}^{N} l_{ij} y_N = 0$. Thus, we recast the network measurement as

$$\zeta_i = \sum_{j=1}^{N} l_{ij} y_j - \sum_{j=1}^{N} l_{ij} y_N = \sum_{j=1}^{N} l_{ij}\mathfrak{e}_{j,N}$$

Let $\sigma_i \triangleq \zeta_i - \zeta_N$ and $\mathfrak{e}_{\boldsymbol{N}} \triangleq \mathrm{col}\{\mathfrak{e}_{i,N}\}$ for $i \in \mathcal{S}_1$ where $\mathcal{S}_1 = \{1, 2, \cdots, \bar{N}\}$. Thus, one may find

$$\sigma_i = \sum_{j=1}^{\bar{N}} l_{ij}^* \mathfrak{e}_{j,N}$$

where $l_{ij}^* = l_{ij} - l_{Nj}, j \in \mathcal{S}$. Let $\bar{L} \triangleq [l_{ij}^*] \in \mathbb{R}^{\bar{N}\times N}$ be obtained by removing the last row of $L - \mathbf{1}\,l_N^{\mathrm{T}}$ where l_k^{T} denotes the k th row of L. Let $L^* \triangleq [l_{ij}^*] \in \mathbb{R}^{\bar{N}\times\bar{N}}$ be the reduced Laplacian which is found by discarding the last column of $\bar{L}$. According to Yang et al. (2011a), $L^* > 0$. Therefore, defining $\boldsymbol{\sigma} \triangleq \mathrm{col}\{\sigma_i\}$, we obtain

$$\boldsymbol{\sigma} = (L^* \otimes \mathcal{I}_p)\mathfrak{e}_{\boldsymbol{N}}$$

In view of (A.15),

$$\boldsymbol{\sigma} = \epsilon\,(\bar{L} \otimes C)(s\mathcal{I} - \mathbb{A})^{-1}\mathbb{B}\boldsymbol{w}$$

Hence,

$$\mathfrak{e}_{\boldsymbol{N}} = \epsilon\,(L^* \otimes \mathcal{I}_p)^{-1}(\bar{L} \otimes C)(s\mathcal{I} - \mathbb{A})^{-1}\mathbb{B}\boldsymbol{w}$$

It shows that $T_{w\mathfrak{e}}^{i,j}$ depends on ϵ, and $\|T_{w\mathfrak{e}}^{i,j}(s)\|_\infty < \epsilon\gamma'$ is attained for some $\gamma' > 0$.

B Proof of Lemma 2

Before starting the proof of Lemma 2, we recall the following result from Sannuti and Saberi (1987).

Lemma 4 (Sannuti and Saberi 1987) *Consider an invertible system which has no invariant zeros and is of uniform rank n_q (i.e. all infinite zeros have the same order n_q). It is described as*

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \\ z = \bar{C}\bar{x} \end{cases}$$

where $\bar{x} \in \mathbb{R}^{pn_q}$ and $z, \bar{u} \in \mathbb{R}^p$. There exist a nonsingular state transformation Γ_0 and a nonsingular input transformation M such that $x = \Gamma_0\,\bar{x}$ and $u = M\bar{u}$ transform the system into

$$\begin{cases} \dot{x} = Ax + B(u + Rx) & \text{(B.2a)} \\ z = Cx & \text{(B.2b)} \end{cases}$$

where A, B, and C are as (6). Also, $R \in \mathbb{R}^{p\times pn_q}$. □

As stated in Section 4.4, the design procedure is three-step. Step 1: According to (Saberi and Sannuti, 1988), the squaring-down pre-compensator for the right-invertible system is given by

$$\Sigma_{1,i} : \begin{cases} \dot{p}_{1,i} = A_{p_{1,i}} p_{1,i} + B_{p_{1,i}} u_{p_{1,i}} & \text{(B.3a)} \\ \bar{u}_i = C_{p_{1,i}} p_{1,i} + D_{p_{1,i}} u_{p_{1,i}} & \text{(B.3b)} \end{cases}$$

where $u_{p_{1,i}} \in \mathbb{R}^p$, and $A_{p_{1,i}}$ is Hurwitz. It makes agent i invertible while adding new stable invariant zeros. Step 2: The rank-equalizing pre-compensator is designed based on (Saberi et al., 1990) and is given by

$$\Sigma_{2,i} : \begin{cases} \dot{p}_{2,i} = A_{p_{2,i}} p_{2,i} + B_{p_{2,i}} u_{p_{2,i}} & \text{(B.4a)} \\ u_{p_{1,i}} = C_{p_{2,i}} p_{2,i} + D_{p_{2,i}} u_{p_{2,i}} & \text{(B.4b)} \end{cases}$$

Defining $p_i = \text{col}\{\bar{x}_i, p_{1,i}, p_{2,i}\}$, the cascade interconnection of $\Sigma_{2,i}$, $\Sigma_{1,i}$ and agent i can be shown by

$$\begin{cases} \dot{p}_i = \Lambda_{p,i} p_i + \Lambda_{u,i} u_{p_{2,i}} + \Lambda_{w,i} \bar{w}_i & \text{(B.5a)} \\ y_i = \begin{bmatrix} C_i & 0 & 0 \end{bmatrix} p_i & \text{(B.5b)} \end{cases}$$

which is an invertible system, with uniform rank n_q. Step 3: According to Sannuti and Saberi (1987), there exist nonsingular state and input transformations such that $p_i = \Gamma_{1,i}\,\text{col}\{x_{a,i}, x_i\}$ and $u_{p_{2,i}} = \Gamma_{2,i} u_{d,i}$ transform (B.5) into the special coordinate basis (s.c.b)

$$\dot{x}_{a,i} = A_{a,i} x_{a,i} + L_{ad,i} y_i + E_{a,i} \bar{w}_i \quad \text{(B.6a)}$$

$$\dot{x}_i = A x_i + B(u_{d,i} + R_{a,i} x_{a,i} + R_{d,i} x_i) + E_{d,i} \bar{w}_i \quad \text{(B.6b)}$$

$$y_i = C x_i \quad \text{(B.6c)}$$

where $x_{a,i} \in \mathbb{R}^{n_{a,i}}$ and $x_i \in \mathbb{R}^{p n_q}$ represent the zero dynamics and the infinite-zero structure, respectively. A, B and C are given by (6). Obviously, one can find

$$E_{o,i} \triangleq \begin{bmatrix} E_{a,i} \\ E_{d,i} \end{bmatrix} = \Gamma_{1,i}^{-1} \Lambda_{w,i}$$

From now on, the goal is to make the system equations (B.6b) similar to (10). The is achieved by means of a feedback to decouple the zero dynamics from x_i subsystem and add the required terms. Therefore, we need to estimate p_i. The measurement available for the system (B.5) is $y^*_{m,i} \triangleq C^*_{m,i} p_i = \text{diag}\{C_{m,i}, \mathcal{I}, \mathcal{I}\} p_i$. Since $(A, C_{m,i})$ is detectable, the pair $(\Lambda_{p,i}, C^*_{m,i})$ is detectable, and one can design an observer to reconstruct p_i by reading $y^*_{m,i}$ and $u_{p_{2,i}}$. Let the estimation error be $\tilde{x}_i \triangleq \text{col}\{x_{a,i}, x_i\} - \text{col}\{\hat{x}_{a,i}, \hat{x}_i\}$ where $\text{col}\{\hat{x}_{a,i}, \hat{x}_i\}$ is the estimated signal. The dynamic equation of $\tilde{x}_i$ is given by (11a) where H_i is Hurwitz stable. Now, we choose the following pre-feedback

$$u_{d,i} = u_i - R_{a,i} \hat{x}_{a,i} - R_{d,i} \hat{x}_i + R \hat{x}_i \quad \text{(B.7)}$$

Considering $u_i = M u'_i$ and substituting (B.7) in (B.6b) give rise to (10) and (11). Thus, $W_i = [R_{a,i},\ (R_{d,i} - R)]$. According to Fig. 2, u'_i is the new input of agent i.

C Proof of Theorem 3

It follows from Lemma 2 that there exists a dynamic compensator that makes agent $i \in \mathcal{S}$ have the dynamics of (5) for an arbitrary R and nonsingular M. $R \in \mathbb{R}^{p \times p n_q}$ is partitioned as $R = [R_1,\ \bar{R}]$ where $R_1 \in \mathbb{R}^{p \times p}$. The vector $\bar{f}_i \in \mathbb{R}^{p n_q}$ for $i \in \mathcal{S}$ is formed as

$$\bar{f}_i = \begin{bmatrix} f_i \\ 0 \end{bmatrix} \Rightarrow f_i = C \bar{f}_i \quad \text{(C.1)}$$

The state error of formation is then denoted $x_{f,i} = x_i - \bar{f}_i$. In view of $A\bar{f}_i = 0$, $R\bar{f}_i = R_1 f_i$ and $\dot{f}_i = 0$, (5) is recast as

$$\dot{x}_{f,i} = A x_{f,i} + B(u_{f,i} + R x_{f,i}) + E_i w_i \quad \text{(C.2a)}$$

$$y_{f,i} = C x_{f,i} \quad \text{(C.2b)}$$

where $u_{f,i} = u_i + R_1 f_i$. In compliance with Section 4, the observer-based protocol for (C.2) will take the following form

$$\dot{\hat{x}}_i = A\hat{x}_i + B(u_{f,i} + R\hat{x}_i) - \epsilon^{-1} K \sum_{j=1}^{N} l_{ij} C \tilde{x}_j \quad \text{(C.3a)}$$

$$u_{f,i} = \epsilon^{-n_q} F S \hat{x}_i \quad \text{(C.3b)}$$

in which $\tilde{x}_j = x_{f,j} - \hat{x}_j$. It then yields closed-loop equations similar to those given in Section 4.3. The rest of the proof is akin to the proof of Lemma 1.

D Proof of Proposition 1

We choose $P_1 = (-P \otimes K_1^{-1})$ in which $P > 0$ is diagonal as $P = \text{diag}\{p_1, \cdots, p_{\bar{N}}\}$ where $p_i \in \mathbb{R}$ must be chosen appropriately. We seek P_1 and $Q_1 > 0$ which satisfy (A.9). Substitution of P_1 in (A.9) results in

$$P_1(\Delta \otimes K_1) + (\Delta \otimes K_1)^{\mathrm{H}} P_1 = -(P\Delta + \Delta^{\mathrm{H}} P) \otimes \mathcal{I}$$

Choosing $Q_1 = Q \otimes \mathcal{I}$, the objective is reduced to show that

$$P\Delta + \Delta^{\mathrm{H}} P = Q$$

where $Q > 0$. We intend to find P so that $\forall v \in \mathbb{R}^{\bar{N}}, v \neq 0$, $v^{\mathrm{T}} Q v > 0$. Since Δ is in the Jordan form, $v^{\mathrm{T}} Q v$ can be expressed as

$$v^{\mathrm{T}} Q v = \sum_{i=1}^{N-1} \text{Re}\{\lambda_i\} p_i v_i^2 + 2 \sum_{i=1}^{N-2} \rho_i p_i v_i v_{i+1} \quad \text{(D.1)}$$

in which $\rho_i \in \{0, 1\}$ and $\rho_i = 1$ if λ_i is a repeated eigenvalue of L. Let $\rho_i = 1$; then one may write:

$$\begin{aligned} v^{\mathrm{T}} Q v &= \frac{1}{3} \sum_{i=1}^{N-1} \text{Re}\{\lambda_i\} p_i v_i^2 \\ &+ \frac{1}{3} \text{Re}\{\lambda_1\} p_1 v_1^2 + \frac{1}{3} \text{Re}\{\lambda_{N-1}\} p_{N-1} v_{N-1}^2 \\ &+ \sum_{i=1}^{N-2} \left(\sqrt{\frac{1}{3} \text{Re}\{\lambda_{i+1}\} p_{i+1}}\, v_{i+1} + \frac{p_i}{\sqrt{\frac{1}{3} \text{Re}\{\lambda_{i+1}\} p_{i+1}}} v_i \right)^2 \\ &+ \sum_{i=1}^{N-2} \left(\frac{1}{3} \text{Re}\{\lambda_i\} p_i - \frac{p_i^2}{\frac{1}{3} \text{Re}\{\lambda_{i+1}\} p_{i+1}} \right) v_i^2 \quad \text{(D.2)} \end{aligned}$$

Obviously, if $\rho_i = 0$, any positive p_i satisfies (A.9) for $Q > 0$. Equation (D.2) will be positive if we set $p_{N-1} = 1$ and define p_i's recursively according to

$$p_i = \frac{\beta^2}{9} p_{i+1} \tag{D.3}$$

for $i \in \{1, \cdots, N-2\}$. Clearly, the first three lines of (D.2) are positive for $p_i > 0, i \in 1, \cdots, \bar{N}$. We show that the last line is positive for this particular choice. In view of (D.3), we have

$$\frac{1}{3}\operatorname{Re}\{\lambda_i\}p_i - \frac{p_i^2}{\frac{1}{3}\operatorname{Re}\{\lambda_{i+1}\}p_{i+1}} = \frac{1}{3}\frac{\beta^2}{9}p_{i+1}\left(\operatorname{Re}\{\lambda_i\} - \frac{\beta^2}{\operatorname{Re}\{\lambda_{i+1}\}}\right) > 0$$

because $\operatorname{Re}\{\lambda_i\} > \beta$. Thus, $P > 0$ is bounded and

$$\|P\| = \max\{1, \left(\tfrac{\beta^2}{9}\right)^{\bar{N}-1}\} \Rightarrow \|P_1\| = \|P\|\|K_1^{-1}\|$$

Moreover, the proposed construction turns out that $\|Q\|$ is bounded from below since

$$v^{\mathrm{T}}Qv > \frac{1}{3}\sum_{i=1}^{N-1}\beta p_i v_i^2 \Rightarrow Q > \frac{1}{3}\beta P$$

It means that $\|Q\| = \|Q_1\| > 4\mathfrak{q}$ where $\mathfrak{q} = \mathcal{O}(\beta)$.

E Simulation Data: Models of Agents

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_1^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2^{\mathrm{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, C_3^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, C_4^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The disturbances act on agents through the following disturbance matrices.

$$E_1^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad E_2^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$
$$E_3^{\mathrm{T}} = \begin{bmatrix} 3 & 1 & 2 & 0 \end{bmatrix}, \qquad E_4^{\mathrm{T}} = \begin{bmatrix} 4 & 4 & 3 & 5 & 3 \end{bmatrix}$$

F Simulation Data: Shaping Procedure

F.1 Squaring-Down Pre-compensators

There is no need for squaring down agents 1 and 2; to keep the coherency we show

$$\Sigma_{1,1} : \bar{u}_1 = u_{p_{1,1}}, \quad \Sigma_{1,2} : \bar{u}_2 = u_{p_{1,2}}$$

and $\Sigma_{1,3}$ and $\Sigma_{1,4}$ are designed for agents 3 and 4:

$$\Sigma_{1,3} : \begin{cases} \dot{p}_{1,3} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} p_{1,3} + \begin{bmatrix} 172.1 \\ -165.1 \end{bmatrix} u_{p_{1,3}} \\ \bar{u}_3 = \begin{bmatrix} 0.1964 & 0.2411 \\ 1 & 1 \end{bmatrix} p_{1,3} + \begin{bmatrix} -1 \\ 16 \end{bmatrix} u_{p_{1,3}} \end{cases}$$

$$\Sigma_{1,4} : \begin{cases} \dot{p}_{1,4} = -p_{1,4} + 5u_{p_{1,4}} \\ \bar{u}_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} p_{1,4} + \begin{bmatrix} -1 \\ -15 \end{bmatrix} u_{p_{1,4}} \end{cases}$$

F.2 Rank-Equalizing Pre-compensators

Agent 1 does not need rank equalization. Thus,

$$\Sigma_{2,1} : u_{p_{1,1}} = u_{p_{2,1}}$$

Since agent 2 and 3 are of relative degree 1, the compensators are the same, obviously with different inputs and outputs; so, we just show $\Sigma_{2,2}$ for agent 2:

$$\Sigma_{2,2} : \begin{cases} \dot{p}_{2,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} p_{2,2} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{p_{2,2}} \\ u_{p_{1,2}} = \begin{bmatrix} 1 & 0 \end{bmatrix} p_{2,2} \end{cases}$$

The compensator for agent 4 is a single intergrator

$$\Sigma_{2,4} : \begin{cases} \dot{p}_{2,4} = u_{p_{2,4}} \\ u_{p_{2,4}} = p_{2,4} \end{cases}$$

F.3 Pre-feedback

Pre-feedback laws can be developed by designing observers for each compensated agent and the following information according to (B.7).

$$R_{d,1} = 0 \qquad R_{a,1} = 0$$

$$R_{d,2} = \begin{bmatrix} -4, \ 2, \ 2 \end{bmatrix} \qquad R_{a,2} = 4$$

$$\begin{aligned} R_{d,3} &= \begin{bmatrix} 80, \ -24, \ 6 \end{bmatrix} \\ R_{a,3} &= \begin{bmatrix} -1, \ 0.70, \ 0.78, \ -1.45, \ -4.36 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E_{d,4} &= \begin{bmatrix} 72, \ -25, \ 7 \end{bmatrix} \\ E_{a,4} &= \begin{bmatrix} -1.00, \ 1.50, \ 1.50, \ -4.50 \end{bmatrix} \end{aligned}$$