Three-stage Filter for Position and Velocity Estimation from Long Baseline Measurements with Unknown Wave Speed

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Abstract— In this paper, an inertial navigation system (INS) aided by a long baseline (LBL) ranging system is presented and shown to have globally exponentially stable (GES) error dynamics. It uses a novel formulation for relating pseudo-range and pseudo-range-rate measurements linearly to position, velocity, and a multiplicative bias parameter, the latter parameter to correct for unknown wave speed, i.e. propagation speed in the medium.

I. INTRODUCTION

In navigation, trilateration of ranges is an important technique for many applications. Common examples of use include Global Navigation Satellite System (GNSS) for all applications above the surface, radar systems for aerial vehicles, and underwater long baseline (LBL) systems for locating submarines, divers, ROVs, and AUVs. Common for all these systems is that they use time-of-arrival (TOA) measurements in order to retrieve the range from multiple transmitters at known locations to a receiver. These range measurements are normally corrupted by uncertainties resulting in biased measurements. The biased measurements are denoted *pseudo-ranges* in order to separate them from the actual ranges. Normally, four pseudo-range measurements are needed to extract the position of the receiver and bias parameter, assuming the bias is common for all measurements. This generally yields two solutions, which is an ambiguity that must be resolved. Having more pseudo-range measurements available typically leads to a unique solution [1], [2]. When range-rate measurements are also corrupted by the bias parameter, they are denoted *pseudo-range-rates*.

In INS, inertial sensors such as gyroscopes and accelerometers are typically used in a dead-reckoning fashion in order to estimate the change in position, velocity, and attitude of a vehicle. The integration of angular velocity and acceleration with noise or bias will lead to an unbounded accumulated error in the estimated states. This error can be corrected through pseudo-range and pseudo-range-rate measurements, e.g. as done in many GNSS applications or as done by Batista [9] and Batista, Silvestre, and Oliveira [10], [11]. The underwater state-of-the-art commercial INS aided by hydro-acoustic measurements is the HAIN system, owned by Konsberg [13].

This paper considers translational motion observers (TMOs) for estimation of position and velocity for under-water vehicles. Under water, GNSS cannot be used, because of the high attenuation rate of electromagnetic waves in water. Therefore, an LBL ranging system is assumed to offer pseudo-ranges and pseudo-range-rates from time-of-arrival (TOA) measurements retrieved with a request-respond strategy from receiver to LBL-transponders. In this set up, the uncertainty of unknown speed of sound in the surrounding water, denoted the *wave speed*, affects the measurements. Therefore, this will be corrected for in the TMO. It is assumed that measurements of acceleration in the inertial frame are available, e.g. gathered from an accelerometer and Attitude Heading Reference System(AHRS).

When designing a TMO for integrating pseudo-range and pseudo-range-rate and inertial measurements, there are two design philosophies: loosely and tightly coupled integration. In a loosely coupled scheme the position, velocity, and bias parameter are explicitly extracted from the pseudo-range and pseudo-range-rate measurements, yielding a linear relation between measurements and states that is exploited when integrating with inertial measurements. A tightly coupled scheme generally uses the raw range measurements and a non-linear measurement model in the integration. The advantage of tight coupling is higher accuracy, while loose coupling leads to a linear measurement model. The latter simplifies the TMO's structure.

The most common example of navigation aided by pseudo-range measurements is GNSS. For practical reasons, GNSS can not rely on a request-respond strategy for TOA measurements. Therefore, the TOA measurements suffer from an additive clock offset between the user and satellite clock. Bancroft [1] and Chaffee and Abel [2] give formulations for the explicit extraction of position and clock bias from the GNSS pseudo-range measurements. This allows for a loose integration scheme, which is used in e.g. Grip et al. [6], [7]. The linear Kalman filter (KF) is the typically used estimator in loosely coupled schemes, because of the linear nature of the translational motion dynamics and measurement equations. In tightly coupled schemes, the highly non-linear nature of the pseudo-range measurement equations need to be accounted for. Therefore, different varieties of non-linear KFs such as the extended KF (EKF) and unscented KF (UKF), in addition to particle filters (PF), are widely used. Tight coupling allowed Hegrenæs et al. [12] to develop an INS aided by only range measurement, in this case from an underwater transponder. Tightly coupled schemes also include formulations where "new" measurements are constructed from the pseudo-range measurements, which leads to a linear relationship between the constructed mea-

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surements and the estimated states. This is done in e.g. Johansen and Fossen [3], and Johansen, Fossen and Goodwin [4] with a similar formulation as the one given in Bancroft [1] and Chaffee and Abel[2]. Both the loosely and tightly coupled estimators can be used with an external attitude observer, e.g. one from Mahoney, Hamel, and Pflimlin [8], or in cooperation with an attitude observer. In the latter, the TMO feedbacks an estimate of the specific force to the attitude observer which the attitude observer compares with the accelerometer measurements, as in [3].

The three-stage filter, a new technique employing tightly coupled schemes was presented in [4]. The first stage constructs "new" measurements, as described in the previous section, in order to achieve a system on a linear time-varying (LTV) form. In the second stage, the LTV system is used by a LTV KF with uniformly globally asymptotically stable (UGAS) error dynamics. The construction of new measurements include non-linear operations that may significantly increase the noise levels, thus lessening the accuracy of the estimate. Near optimal performance can be recovered in the third stage where a linearised KF is used, linearising the original system about the state estimate from the LTV KF in stage two, which is shown to achieve UGAS error dynamics of the origin. This strategy yields UGAS error dynamics of the origin of the cascade of KFs, while maintaining high accuracy in the estimate.

In this paper, the three-stage filter from [4] is employed for estimating an underwater vehicle's position and velocity with an INS aided by an LBL system with unknown wave speed. Unknown wave speed leads to a multiplicative bias as opposed to the additive bias in [4]. This demands a formulation relating the pseudo-range and pseudo-range-rate measurements to position, velocity, and bias parameter. A formulation relating pseudo-range to position and bias was developed by [9], but a linear relationship was not achieved since the state space was augmented to deal with the non-linearities instead. In order to relate the states linearly to the measurements, the method of [1] and [2] is tailored for multiplicative bias and extended to the case with pseudo-range-rate measurements.

The contribution of this paper is the derivation of a filter with GES error dynamics. This filter uses a novel formulation relating the position, velocity, and multiplicative bias linearly to the pseudo-range and pseudo-range-rate measurements.

This paper is organised as follows: In Section II, the constructed measurements that has a linear relationship to the states are derived for both minimal and redundant measurements. In Section III, the three-stage filter is stated, and the origin of the filter is proven GES. The implementation and simulation of the filters follow in Section IV, before the results are discussed in Section V.

II. MEASUREMENT MODEL

In this section, the notation is first defined and measurement equations derived in Section II-A before the constructed measurement equations are derived in Section II-B.

A. Measurement Equations

From the LBL system used in this paper, pseudo-range and pseudo-range-rate from transponder i are found from TOA and Doppler shift measurements, denoted t_i and Δ_i, respectively. $y_i = c_0 t_i$ and $\nu_i = c_0 \Delta_i$ are the pseudo-range and pseudo-range-rates, respectively, and c_0 denotes the guessed wave propagation speed. It deviates with a factor of $\sqrt{\beta}$ from the actual wave propagation speed, i.e. $c_0 \sqrt{\beta} = c$. Let p^n and v^n denote the position and velocity of the vehicle, and $\breve{p}_i^n$ the placement of transponder i in the North-East-Down (NED) coordinate frame. Denoting ρ_i and $\dot{\rho}_i$ the true range and range-rate from transponder i, we recognise

$$\rho_i = \|p^n - \breve{p}_i^n\| \tag{1}$$

$$\dot{\rho}_i = \frac{(p^n - \breve{p}_i^n)^\top v^n}{\|p^n - \breve{p}_i^n\|} \tag{2}$$

where $\|\cdot\|$ is the 2-norm. This allows us to write

$$y_i = \frac{1}{\sqrt{\beta}}(\rho_i + \epsilon_{y,i}) = \frac{1}{\sqrt{\beta}}(\|p^n - \breve{p}_i^n\| + \epsilon_{y,i}) \tag{3a}$$

$$\nu_i = \frac{1}{\sqrt{\beta}}(\dot{\rho}_i + \epsilon_{\nu,i}) = \frac{1}{\sqrt{\beta}}\left(\frac{(p^n - \breve{p}_i^n)^\top v^n}{\rho_i} + \epsilon_{\nu,i}\right) \tag{3b}$$

where $\epsilon_{y,i}$ and $\epsilon_{\nu,i}$ are zero-mean, Gaussian measurement noise with standard deviations σ_y and σ_ν, respectively.

B. Constructed Measurement

In the following derivations, the noises $\epsilon_{y,i}$ and $\epsilon_{\nu,i}$ are not explicitly included for simplicity of presentation.

Constructing the measurement

$$y_i^2 = \frac{1}{\beta}(p^n - \breve{p}_i^n)^\top(p^n - \breve{p}_i^n)$$

and expanding and rearranging yields

$$\beta y_i^2 = r - 2\breve{p}_i^{n\top}p^n + \|\breve{p}_i^n\|^2 \tag{4}$$

where $r = p^{n\top}p^n$.

4 transponders: We define a partial state vector and a selection matrix

$$x = \begin{bmatrix} p^n \\ \beta \end{bmatrix}, \ M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

such that r can be expressed in terms of x as

$$r = x^\top M x \tag{5}$$

We notice that (4) and (5) adds up to 5 equations for the 5 unknowns p^n, β, and r. Now, r has to be found explicitly, and we concatenate (4)

$$C_y x - rl = z \tag{6}$$

$$C_y = \begin{bmatrix} C_{yp} & C_{y\beta} \end{bmatrix}, \ C_{yp} = \begin{bmatrix} 2\breve{p}_1^{n\top} \\ \vdots \\ 2\breve{p}_4^{n\top} \end{bmatrix} C_{y\beta} = \begin{bmatrix} y_1^2 \\ \vdots \\ y_4^2 \end{bmatrix}$$

where $l = [1, 1, 1, 1]^\top$ and $z = [\|\breve{p}_1^n\|^2, ..., \|\breve{p}_4^n\|^2]^\top$. If C_y is invertible, we can find

$$x = r\underbrace{C_y^{-1}l}_{c} + \underbrace{C_y^{-1}z}_{w} = rc + w \quad (7)$$

Inserting (7) into (5) yields the second order equation

$$r = (rc + w)^\top M(rc + w) = r^2 c^\top Mc + 2rc^\top Mw + w^\top Mw$$
$$r^2 c^\top Mc + r\underbrace{(2c^\top Mw - 1)}_{h} + w^\top Mw = 0$$

This equation has two solutions

$$r_{1,2} = \frac{-h \pm \sqrt{h^2 - 4c^\top Mc \cdot w^\top Mw}}{2c^\top Mc} \quad (8)$$

where one is the correct, denoted r_1, and the other is wrong, r_2. In order to solve this ambiguity, one might often use circumstantial knowledge, such as that x, when r_2 is inserted into (7), for example has a position that is below the sea floor or a β that greatly differs from 1.

5 or more transponders: With five or more transponders, r can be eliminated by differencing the constructed equations (4). We are now left with at least 4 linear equations for the 4 unknowns p^n and β and can disregard the non-linear equation (5).

$$\beta(y_i^2 - y_m^2) = -2(\breve{p}_i^n - \breve{p}_m^m)^\top p^n + \|\breve{p}_i^n\|^2 - \|\breve{p}_m^n\|^2 \quad (9)$$

Concatenating (9) for $i \in [1, m-1]$ yields

$$C_y' x = [C_{yp}', C_{y\beta}']x = z' \quad (10)$$

$$C_{yp}' = \begin{bmatrix} 2(\breve{p}_1^n - \breve{p}_m^n)^\top \\ \vdots \\ 2(\breve{p}_{m-1}^n - \breve{p}_m^n)^\top \end{bmatrix}, C_{y\beta}' = \begin{bmatrix} y_1^2 - y_m^2 \\ \vdots \\ y_{m-1}^2 - y_m^2 \end{bmatrix} \quad (11)$$

where $z' = [\|\breve{p}_1^n\|^2 - \|\breve{p}_m^n\|^2, ..., \|\breve{p}_{m-1}^n\|^2 - \|\breve{p}_m^n\|^2]^\top$.

Here, we have related the range measurements to the partial state, x, linearly. Now, pseudo-range-rate measurements have to be related to the full state. The pseudo-range-rate is given in (3b) and we see that the formulation

$$\beta(y_i \nu_i - y_m \nu_m) = -(\breve{p}_i^n - \breve{p}_m^n)^\top v^n$$

leads to a formulation that is linear in both β and v^n. Concatenating the above equation yields

$$\underbrace{\begin{bmatrix} y_1 \nu_1 - y_m \nu_m \\ \vdots \\ y_{m-1}\nu_{m-1} - y_m \nu_m \end{bmatrix}}_{C_{\nu\beta}} \beta + \underbrace{\begin{bmatrix} (\breve{p}_1^n - \breve{p}_m^n)^\top \\ \vdots \\ (\breve{p}_{m-1}^n - \breve{p}_m^n)^\top \end{bmatrix}}_{C_{\nu v}} v^n = 0 \quad (12)$$

This is valid for $m \geq 4$. Depending on whether $m = 4$ or $m \geq 5$, (12) can be stacked with either (6) or (10). This
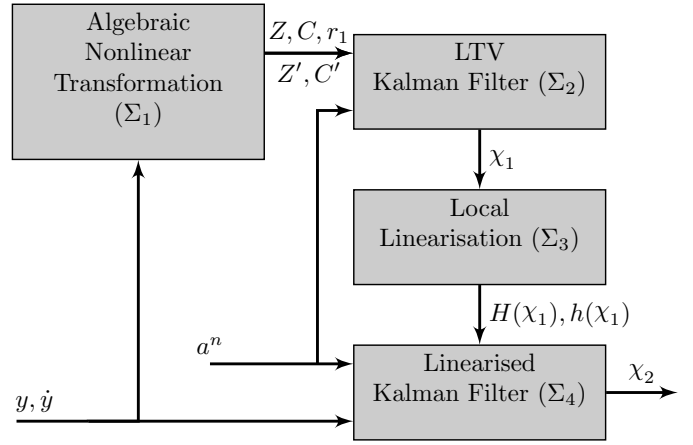

Fig. 1.   Block diagram showing the three-stage filter. The three stages naming the filter are Σ_1, Σ_2, and Σ_4.

yields

$$C(t)\chi - Lr_1 = Z \quad (13)$$

$$\underbrace{\begin{bmatrix} C_{yp} & C_{y\beta} & 0_{3\times3} \\ 0_{3\times3} & C_{\nu\beta} & C_{\nu v} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} p^n \\ \beta \\ v^n \end{bmatrix}}_{\chi} - \underbrace{\begin{bmatrix} l \\ 0_{3\times1} \end{bmatrix}}_{L} r_1 = \underbrace{\begin{bmatrix} z \\ 0_{3\times1} \end{bmatrix}}_{Z}$$

for $m = 4$ and

$$C'(t)\chi = Z' \quad (14)$$

$$\underbrace{\begin{bmatrix} C_{yp}' & C_{y\beta}' & 0_{3\times3} \\ 0_{3\times3} & C_{\nu\beta} & C_{\nu v} \end{bmatrix}}_{C'} \chi = \underbrace{\begin{bmatrix} z' \\ 0_{3\times1} \end{bmatrix}}_{Z'}$$

for $m \geq 5$. Note that the constructed measurement equation is linearly time-varying in $C(t)$ or $C'(t)$.

To the best of the authors knowledge, this formulation is a contribution to the field, being a simple extension of the work of [9] and [1].

III. POSITION AND VELOCITY FILTERS

The structure of the three-stage filter is shown in Figure 1. It contains four subsystems that are explained in this section.

A. Subsystem Σ_1: Non-linear Algebraic Transform

Subsystem Σ_1 is given by the non-linear algebraic transform (13) or (14) in Section II-B, relating the pseudo-range and pseudo-range-rate measurements linearly to the state vector, χ.

B. Subsystem Σ_2: LTV Kalman-Bucy filter

Subsystem Σ_2 is an LTV Kalman-Bucy filter given by the system model

$$\dot{p}^n = v^n$$
$$\dot{\beta} = \varepsilon_\beta$$
$$\dot{v}^n = a^n(t) + \varepsilon_a$$

or

$$\dot{\chi} = \underbrace{\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \chi + \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}}_{B} a^n(t) + \underbrace{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \varepsilon_\beta \\ \varepsilon_a \end{bmatrix}}_{\varepsilon} \quad (15)$$

with measurement (13) or (14). The state estimate of this KF is denoted χ_1. $a^n(t)$ is assumed to be a known, bounded signal, and ϵ_β and ϵ_a defines the process noise covariance matrix Q.

The non-linear operations performed in the construction of (13) or (14), and the use of (8) or disregarding (5), may significantly alter the noise characteristics from the original measurements (3a)–(3b). Thus, a new measurement covariance matrix, $R_1(t)$, has to be found. This is outlined in Appendix A.

C. Subsystem Σ_3: Local Linearisation

Subsystem Σ_3 linearises the original measurement equations (3a) and (3b) about the state estimate from Σ_2 and feed-forwards it to subsystem Σ_4.

Let $h(\chi) \in \mathcal{R}^{2m}$ denote the measurement function vector whose elements $h_i(\chi)$ are given by (3a) and $h_{i+m}(\chi)$ are given by (3b) for $i \in [1, m]$. The linearised measurement matrix becomes

$$H(\chi_1) = \begin{bmatrix} \frac{dh_1(\chi)}{d\chi} \\ \vdots \\ \frac{dh_{2m}(\chi)}{d\chi} \end{bmatrix}\Bigg|_{\chi=\chi_1} \in \mathcal{R}^{2m \times 7} \quad (16)$$

A first-order Taylor approximation of the measurement function is found as

$$h(\chi) \approx h(\chi_1) + H(\chi_1)(\chi - \chi_1) \quad (17)$$

Both $H(\chi_1)$ and $h(\chi_1)$ are feed-forwarded to subsystem Σ_4.

D. Subsystem Σ_4: Linearised Kalman-Bucy filter

Subsystem Σ_4 is a Kalman-Bucy filter based on (15) with measurements (3a)–(3b), measurement model (17), and measurement matrix (16). χ_2 denotes the state estimate of this KF.

E. Stability Analysis

This section considers the stability of the error dynamics of the cascaded system Σ_1–Σ_4 for the cases where $m = 4$ and $m \geq 5$.

Assumption 1: The matrix $[C_{yp}, C_{p\beta}]$ in (13) or $[C'_{yp}, C'_{p\beta}]$ in (14) has full rank.

Assumption 2: The ambiguity between r_1 and r_2 can be resolved when $m = 4$.

Proposition 1: The origin of the error dynamics of the cascaded system Σ_1–Σ_4 is GES under Assumption 1 and 2.

Proof: For both $m = 4$ and $m \geq 5$, it is trivial to show that $C_{\nu\nu}$ has rank 3 under Assumption 1 since its columns are linear combinations of linearly independent vectors. Also,

the non-linear algebraic transformation Σ_1 has one unique solution for $m \geq 5$. This is true for $m = 4$ as well under Assumption 2.

Now, Johansen and Fossen [5] shows that if the error dynamics of the origins of Σ_2 and Σ_4 are GES, then the origin $\chi_1 = \chi_2 = 0$ of the cascaded error dynamics Σ_1–Σ_4 is GES.

For stability of Σ_2, it is required that the observability Gramian

$$W = \int_{t_0}^{t_1} \Phi^\top(s, t_0) C^\top(s) C(s) \Phi(s, t_0) ds \quad (18)$$

has full rank. Since A is a nilpotent matrix, the state transition matrix can be found exactly as $\Phi(s, t_0) = I_7 + A(s, t_0)h$, where I_7 is the 7×7 identity matrix and h the sampling time. The full rank of $\Phi(s, t_0)$ conserves the rank of $C^\top(s)C(s)$ in (18). Hence, (18) has full rank under Assumption 1.

For the stability of Σ_4, we argue that Σ_4 is founded on the same dynamics as Σ_2 and that the non-linear transformation from the measurement equations (3a)–(3b) of Σ_4 to the measurement equations (13) or (14) of Σ_2 is invertible, hence, the systems Σ_2 and Σ_4 are equivalent. Thus, stability of Σ_2 guarantees stability of Σ_4, which concludes the proof. ∎

Remark 1: Assumption 1 is true when the transponder placement is non-coplanar.

Remark 2: The above proof guarantees that the error dynamics of the origins of Σ_2 and Σ_4 are GES and uses the proof from Johansen and Fossen [5] to show GES of the error dynamics of the origin of the cascade Σ_1–Σ_4.

IV. SIMULATION

In this section, the three-stage filter Σ_1–Σ_4 is tested in simulations and compared to a state-of-the-art EKF. The implementation of simulation and filters is covered in Section IV-A. Further, in Section IV-B, the results of the simulations are shown.

A. Implementation

The simulation set up assumes $m = 4$ hydroacoustic transponders placed non-coplanarly on the sea floor at

$$\breve{p}_1^n = [10, 10, 0]^\top, \ \breve{p}_2^n = [10, -10, 1]^\top$$
$$\breve{p}_3^n = [-10, 10, 2]^\top, \ \breve{p}_4^n = [-10, -10, 0]^\top$$

For comparisons of transient behaviour, there is a large error in the initial values: The true initial position and velocity was $p_0 = [0, 0, 0]^\top$ and $v_0 = [0, 0, 0]^\top$, and the wave speed was set to $c^* = 1450 \ ms^{-1}$. Two different scenarios were simulated: one with a large initial error in order to show the transient behaviours of the filters, and one with no initial error to compare steady-state behaviour.

For the simulation with large initial errors, $\beta = 0.8751$, i.e. $c_0 = 1550 \ ms^{-1}$, was used. The initial state was $\hat{p}_0 = [10, -15, -10]^\top$, $\hat{\beta}_0 = 1$, and $\hat{v}_0 = [0.1, 0.2, 0.1]^\top$.

For the simulation with no initial errors, $\beta = 1$, i.e. $c_0 = 1450 \ ms^{-1}$, was used along with the initial state $\hat{p}_0 = [0, 0, 0]^\top$, $\hat{\beta}_0 = 1$, and $\hat{v}_0 = [0, 0, 0]^\top$.

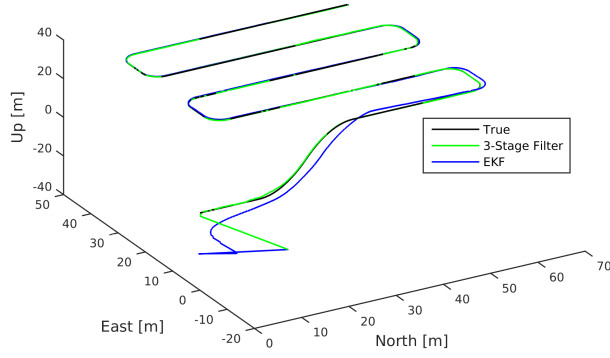Fig. 2.   Real and estimated trajectories for large initial error.

Fig. 3.   Transient behaviour of position estimate.

Range and range-rate measurements were gathered from all transponders with a noise of standard deviation $\sigma_y = 0.10 \ m$ and $\sigma_\nu = 0.05 \ ms^{-1}$ at a frequency of 2 Hz. The inertial acceleration measurements are assumed to be gathered with 100 Hz and have noise with standard deviation $\sigma_v = 0.01 \ ms^{-2}$. This gives a state covariance matrix $Q = \text{diag}(\sigma_\beta, \sigma_v, \sigma_v, \sigma_v)$ for Σ_2, Σ_4, and EKF, and a measurement covariance matrix

$$R = \text{diag}(\sigma_y, ..., \sigma_y, \sigma_\nu, ..., \sigma_\nu)$$

for Σ_4 and EKF. The R-matrix used in Σ_2 is given by (19) in Appendix A. $\sigma_\beta = 10^{-6}$ was determined after tuning. The initial covariance matrices was determined after tuning to be $P(0) = \text{diag}(10, 10, 10, 10^{-3}, 5 \cdot 10^{-2}, 5 \cdot 10^{-2}, 5 \cdot 10^{-2})$

The linearisation in Σ_3 can either be about the a priori estimate $\bar{\chi}_1$ or the a posteriori estimate $\hat{\chi}_1$. It is not obvious which estimate is better, as $\bar{\chi}_1$ is more noisy, but uncorrelated with the measurement, whereas $\hat{\chi}_1$ is less noisy, but correlated with the measurements. This is an item for discussion in Section V.

The simulations were run for 300 seconds, and the filters were updated with a frequency of 100 Hz.

B. Results

In Figure 2 the true trajectory is shown along with the estimated trajectories. In Figures 3-5, the transient behaviours of the three-stage filter and EKF are compared. $\hat{\chi}_1$ is feed-forwarded from the LTV KF to the linearised KF.

V. DISCUSSION

Although the simulations merely provide proof-of-concept and give little evidence of real-life performance, it is still interesting to benchmark the performance of the three-stage filter against an EKF. In Figures 3–5, we see that the three-stage filter converges significantly faster than the EKF for all state values, especially the velocity estimate in Figure 4.

The Euclidean distance from the estimated to the true trajectories is shown in Figure 6, where we see that the filters have similar steady-state performance. The errors of estimated position and velocity are not shown in this case,
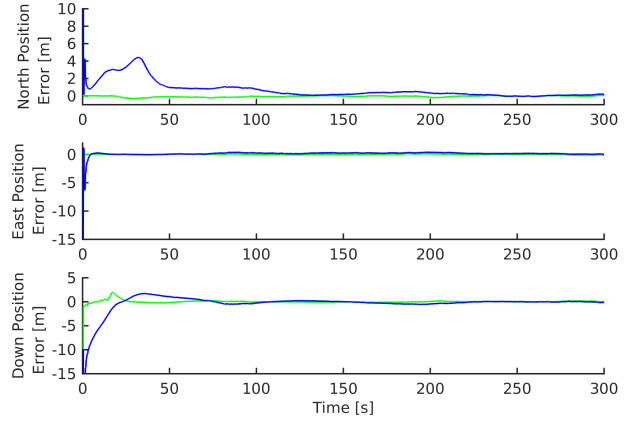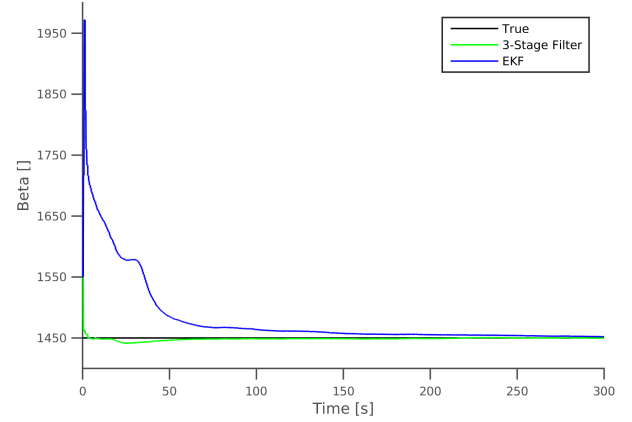

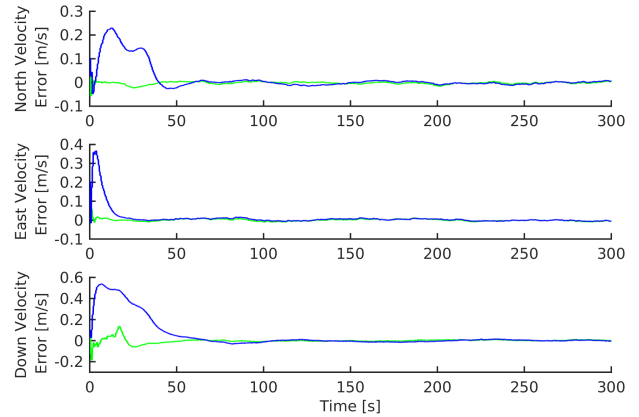Fig. 4.   Transient behaviour of wave speed estimate.

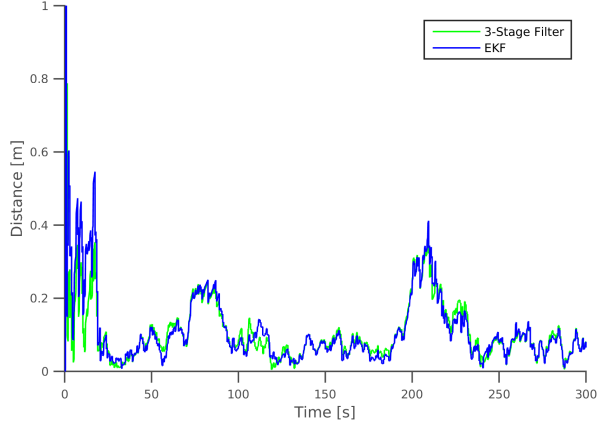Fig. 5.   Transient behaviour of velocity estimate.

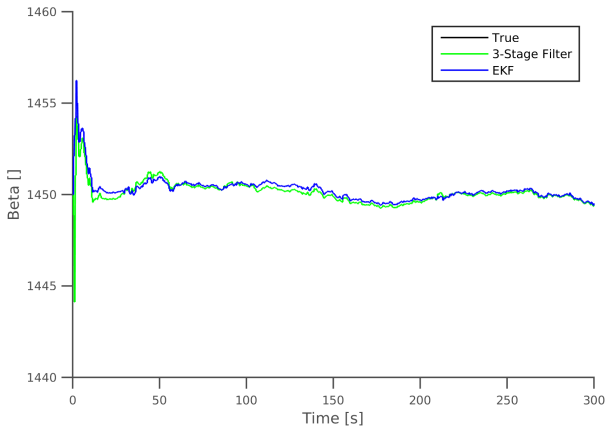Fig. 6. Euclidean distance errors in steady state.

Fig. 7. Steady-state behaviour of wave speed estimate.

because they display a similar performance as Figure 6. Figure 7 confirms this, as it can be seen that the wave speed estimates of the two filters are nearly identical.

It is important to note that in addition to better transient behaviour, three-stage filter also guarantees global exponential convergence, whereas the EKF can diverge because of the linearisation of the covariance update law.

The computing power demanded by the three-stage filter is approximately twice as high as the EKF, which is confirmed in simulations. This is no surprise, as it employs two KFs compared to the EKF's one.

The performance difference between feed-forwarding $\hat{\chi}_1$ and $\bar{\chi}_1$ to Σ_3 are shown in simulations to be indistinguishable. This, however, may not always be the case, especially in applications where there is a stronger correlation between $\hat{\chi}_1(t)$ and $y(t)$.

VI. CONCLUSION

In this paper, an INS aided by LBL measurements for navigation of underwater vehicles with unknown wave propagation speed was derived and proven to have GES error dynamics. A novel formulation that relates pseudo-ranges and pseudo-range-rates linearly to position, velocity, and uncertainty in wave propagation speed, which is a multiplicative bias parameter, was used. This formulation is an extension of Batista [9], Bancroft [1], and Chaffee and Abel [2]. The formulation was used in the three-stage filter[4]. In fact, this was the first time a multiplicative bias-formulation is implemented with the three-stage filter.

The filter was in computer simulations shown to successfully estimate the position, velocity, and bias. It also showed promising performance when benchmarked against an EKF.

This paper inspires further work such as extending the state space to include estimation of specific force, and feed-backing this into an attitude observer. The results in this paper will also be verified using an experimental platform that is currently being built.

APPENDIX

A. Covariance of Constructed Measurements

The constructed measurement equations with noise are

$$y_i^2 = \frac{1}{\beta}(\rho_i + \epsilon_{y,i})^2 = \frac{1}{\beta}(\rho_i^2 + 2\epsilon_{y,i}\rho_i + \epsilon_{y,i}^2)$$

$$y_i^2 - y_m^2 = \frac{1}{\beta}(\rho_i^2 - \rho_m^2$$
$$+ 2\epsilon_{y,i}\rho_i - 2\epsilon_{y,m}\rho_m + \epsilon_{y,i}^2 - \epsilon_{y,m}^2)$$

$$y_i\nu_i - y_m\nu_m = \frac{1}{\beta}((\rho_i + \epsilon_{y,i})(\dot{\rho}_i + \epsilon_{\nu,i})$$
$$- (\rho_m + \epsilon_{y,m})(\dot{\rho}_m + \epsilon_{\nu,m}))$$
$$= \frac{1}{\beta}(\rho_i\dot{\rho}_i - \rho_m\dot{\rho}_m + \epsilon_{y,i}\dot{\rho}_i - \epsilon_{y,m}\dot{\rho}_m$$
$$+ \epsilon_{\nu,i}\rho_i - \epsilon_{\nu,m}\rho_m + \epsilon_{y,i}\epsilon_{\nu,i} - \epsilon_{y,m}\epsilon_{\nu,m})$$

Now, the mean and variance of the constructed measurements can be found and the covariance matrix $R_1(t)$ assembled

$$m_{y,i} = \mathrm{E}(y_i^2 - \frac{1}{\beta}\rho_i^2) = \frac{1}{\beta}\sigma_y^2 = m_y$$

$$m'_{y,i} = \mathrm{E}(y_i^2 - y_m^2 - \frac{1}{\beta}(\rho_i^2 - \rho_m^2)) = 0$$

$$R_{y,i}(t) = \mathrm{Var}(y_i^2) = \frac{2}{\beta^2}\left(2\rho_i^2\sigma_y^2 + \sigma_y^4\right)$$

$$R'_{y,i}(t) = \mathrm{Var}(y_i^2 - y_m^2) = \frac{4}{\beta^2}\left((\rho_i^2 + \rho_m^2)\sigma_y^2 + \sigma_y^4\right)$$

$$m_{\nu,i} = \mathrm{E}(y_i\nu_i - y_m\nu_m - \frac{1}{\beta}(\rho_i\dot{\rho}_i - \rho_m\dot{\rho}_m)) = 0$$

$$R_{\nu,i}(t) = \mathrm{Var}(y_i\nu_i - y_m\nu_m)$$
$$= (\dot{\rho_i}^2 + \dot{\rho_m}^2)\sigma_y^2 + (\rho_i^2 + \rho_m^2)\sigma_\nu^2 + \sigma_y^2\sigma_\nu^2$$

$$R_1(t) = \mathrm{diag}([R_{y,1}(t), ..., R_{y,4}(t), R_{\nu,1}(t), ..., R_{\nu,3}(t)]) \tag{19}$$

$$R'_1(t) = \mathrm{diag}([R'_{y,1}(t), ..., R'_{y,4}(t), R_{\nu,1}(t), ..., R_{\nu,3}(t)])$$

Since the constructed measurement y_i^2 is biased, m_y must be subtracted from y_i^2 before it is used, in the case when $m = 4$. For the above equations, β can be assumed to be approximately 1, or a conservative value of e.g. $\beta = 0.8$ can be used.

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