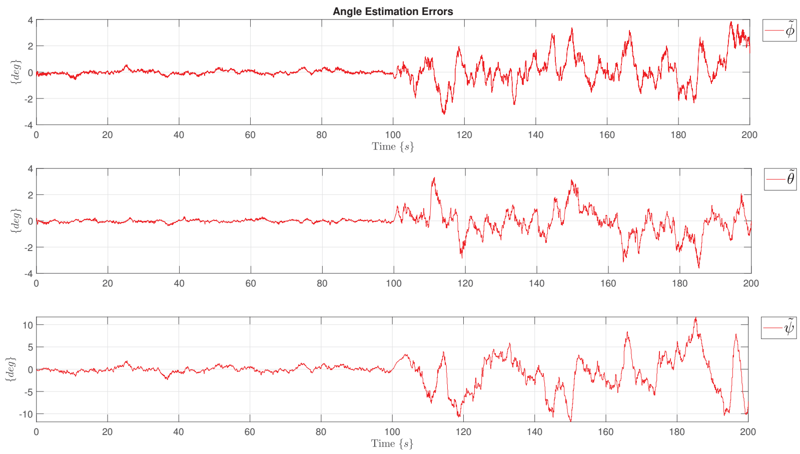

Fig. 6. Plots showing the attitude estimation errors for Case 2.

positive effect on the estimates velocity state, roll- and pitch angle, even though not as great as for heading. Position estimates is about unchanged, which is to be expected as position measurements are still available. Using specific force as a reference vector had next to no impact on accuracy at all. It is a likely speculation that this is because this reference vector is already integrated into the navigation model.

APPENDIX

A. System Observability

For the navigation system equations to converge, all states need to be observable. For this, the previously mentioned position- and velocity measurements are combined attitude measurements. They are handled as loosely coupled measurements while in reality being tightly, but that simplification is justified in Section B. Observability without direct attitude measurements is also possible, but require specific motion that is discussed in [16]

Using theory from [17, Chap. 6], the following theorem proves local observability:

Let $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}(t) \in \mathbb{R}^{m\times n}$ be C^{n-1}. Then the pair $(\boldsymbol{A}(t), \boldsymbol{C}(t))$ is observable at t_0 if there exists a finite $t_1 > t_0$ such that

$$rank \begin{bmatrix} \boldsymbol{N}_1(t_1) \\ \boldsymbol{N}_2(t_1) \\ \vdots \\ \boldsymbol{N}_n(t_1) \end{bmatrix} = n \tag{47}$$

where

$$\boldsymbol{N}_{i+1}(t) = \boldsymbol{N}_i(t)\boldsymbol{A}(t) + \dot{\boldsymbol{N}}_i(t), \quad i = 1, 2 \ldots, n-1 \tag{48}$$

with

$$\boldsymbol{N}_1 = \boldsymbol{C}(t) \tag{49}$$

With a slight change of notation, we use

$$\boldsymbol{C}(t) = \boldsymbol{H}(t) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_3 \end{bmatrix} \tag{50}$$

and that $\boldsymbol{A}(t) = \boldsymbol{F}$, where all the elements of $\boldsymbol{F}$ is known at time t as they build upon previous estimates. This gives that

$$\boldsymbol{N}_o(t) = \begin{bmatrix} \boldsymbol{N}_1(t) \\ \boldsymbol{N}_2(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{0}_3 & -\hat{\boldsymbol{R}} & -\hat{\boldsymbol{R}}\boldsymbol{S}(\hat{f}^b) & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{0}_3 & \boldsymbol{0}_3 & -\boldsymbol{S}(\hat{\boldsymbol{\omega}}) & -\boldsymbol{I}_3 \end{bmatrix} \tag{51}$$

Using the fact that rotation matrices by definition has rank of 3, it is clear that $\boldsymbol{N}_o(t)$ has rank of $n = 15$, meaning that they system is at least locally observable.

B. Attitude Observability from reference vectors

In Section A, the attitude measurements are treated as being loosely coupled, while they in reality are tightly coupled in the form of reference vectors. It is however a reasonable simplification to use, as two or more reference vectors still provide full observability of the attitude.

Proof: The linearized measurement model for reference vectors shows only one non-zero element, that is $\frac{d}{d\boldsymbol{a}}\tilde{\boldsymbol{r}}^n = -\boldsymbol{R}^n_{\hat{b}}\boldsymbol{S}(\hat{\boldsymbol{r}}^b)$. As this matrix is a known rotation of a skew-symmetric matrix, the rank will be the same as the rank of the skew-symmetric matrix. To find the rank of $\boldsymbol{S}(\hat{\boldsymbol{r}}^b)$, its null-space will be inspected. The symbols $\boldsymbol{a}$ and $\boldsymbol{b}$ will for the observability analysis represent two arbitrary three-dimensional vectors. As the notation $\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b}$ is the cross product between the two vectors, this product will only give the null-vector when $\boldsymbol{a} \parallel \boldsymbol{b}$, meaning that the null-space of $\boldsymbol{S}(\boldsymbol{a})$ is one-dimensional and that the matrix' is of rank 2. The attitude of any 3-D rigid body has three degrees of freedom, meaning that an observability matrix from a single reference vector insufficient to observe the complete attitude.

When another reference vector is introduced, the measurement vector's observation of the attitude turns into

$$\frac{d}{d\boldsymbol{a}}\boldsymbol{h}(\boldsymbol{x}) = \frac{d}{d\boldsymbol{a}}\begin{bmatrix}\tilde{\boldsymbol{r}}_1^n \\ \tilde{\boldsymbol{r}}_2^n\end{bmatrix} = \begin{bmatrix}-\boldsymbol{R}_{\hat{b}}^n \boldsymbol{S}(\hat{\boldsymbol{r}}_1^b) \\ -\boldsymbol{R}_{\hat{b}}^n \boldsymbol{S}(\hat{\boldsymbol{r}}_2^b)\end{bmatrix} \tag{52}$$

As the rotation is irrelevant for the rank of this matrix, it will be omitted further observability analysis, and we define

$$\boldsymbol{H}_r = \begin{bmatrix}\boldsymbol{S}(\hat{\boldsymbol{r}}_1^b) \\ \boldsymbol{S}(\hat{\boldsymbol{r}}_2^b)\end{bmatrix} \tag{53}$$

where $\boldsymbol{H}_r \in \mathbb{R}^{6\times 3}$

Again, the easiest way to find the rank of the matrix $\boldsymbol{H}_r$ is to inspect its null-space. It was previously stated that the only way that $\boldsymbol{S}(\hat{\boldsymbol{r}}_1^b)\boldsymbol{a} = \vec{0}$ is if $\hat{\boldsymbol{r}}_1 \parallel \boldsymbol{a}$. Subjecting the matrix $\boldsymbol{H}_r$ to the same conditions, it shows that $\boldsymbol{H}_r\boldsymbol{a} = \vec{0}$ if and only if $\boldsymbol{a} \parallel \hat{\boldsymbol{r}}_1$ and $\boldsymbol{a} \parallel \hat{\boldsymbol{r}}_2$, implying that $\hat{\boldsymbol{r}}_1 \parallel \hat{\boldsymbol{r}}_2$. If the two reference vectors are to be non-parallel, in other words linearly independent, then $\boldsymbol{H}_r$ would have an empty null-space and a guaranteed rank of 3, giving full observability of the attitude.

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